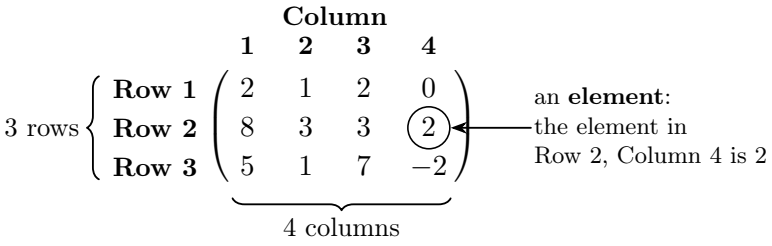
# Matrices

Order, Addition, Scalar Multiplication, Matrix Multiplication and Word Problems

Secondary 4 — G3 Mathematics

## Notes

The **order** of a matrix refers to the dimensions of the matrix. The matrix below has 3 rows and 4 columns, so its order is $3 \times 4$ (read '3 by 4'). The numbers in a matrix are known as its **elements**.




Order of a matrix = (number of rows) × (number of columns) = $3 \times 4$

**Column matrix** A matrix that has only one column, e.g. $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

**Row matrix** A matrix that has only one row, e.g. $\begin{pmatrix} 2 & -1 & 0 \end{pmatrix}$.

**Square matrix** A matrix that has the same number of rows and columns, e.g. $\begin{pmatrix} 5 \end{pmatrix}$ or $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 12 & 2 & 5 \\ 4 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}$.

**Zero (null) matrix** Every element of the matrix is zero. It is denoted by **O**, e.g. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

**Identity matrix (unit matrix)** It is a **square matrix**, denoted by **I**. All the elements in the **principal diagonal** are 1s and the other elements are 0s.

e.g. $\begin{pmatrix} 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (principal diagonal)

Any matrix **A** multiplied by an identity matrix (of a suitable order) remains as **A**, i.e. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

Two matrices are equal if and only if they have the **same order** and their **corresponding elements** are equal.

Two matrices **A** and **B** have to be of the **same order** in order to be added or subtracted. We simply add or subtract the **corresponding elements**.

e.g. $\begin{pmatrix} 2 & 0 \\ 5 & 1 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2+3 & 0+2 \\ 5+1 & 1-2 \\ -2-1 & 5+3 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 6 & -1 \\ -3 & 8 \end{pmatrix}$

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Corresponding elements sit in the same position. In the subtraction below, the boxed elements are in Row 1, Column 2 of each matrix.

$$\underset{2\times 2}{\begin{pmatrix} 2 & \boxed{1} \\ 4 & 0 \end{pmatrix}} - \underset{2\times 2}{\begin{pmatrix} 5 & \boxed{-1} \\ 2 & 3 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} -3 & \boxed{2} \\ 2 & -3 \end{pmatrix}} \quad 1-(-1)=2$$

**Properties**

- Matrix addition is **commutative**: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$.
- Matrix addition is **associative**: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})=\mathbf{A}+\mathbf{B}+\mathbf{C}$.

When a matrix $\mathbf{A}$ is multiplied by a real number $k$, the matrix $k\mathbf{A}$ is obtained by multiplying **each element** in $\mathbf{A}$ by the real number $k$.

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

e.g. $$5\begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5\times 2 & 5\times 4 \\ 5\times 0 & 5\times 3 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 0 & 15 \end{pmatrix}$$

every element is multiplied by 5

Given that $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, the matrix multiplication is

$$\mathbf{AB}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

**Example.** Find $\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix}$.

Each element of the product comes from one row of the 1st matrix and one column of the 2nd matrix: multiply their first elements, their second elements and their third elements, then add. The shaded row and column give the shaded element.

$$\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & \square \\ \square & \square \end{pmatrix}$$

Row 1, Column 1: $2\times 1+4\times 3+1\times 4=18$

$$\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ \square & \square \end{pmatrix}$$

Row 1, Column 2: $2\times 6+4\times 2+1\times 0=20$

$$\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ 26 & \square \end{pmatrix}$$

Row 2, Column 1: $3\times 1+1\times 3+5\times 4=26$

$$\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ 26 & 20 \end{pmatrix}$$

Row 2, Column 2: $3\times 6+1\times 2+5\times 0=20$

$$\therefore \begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ 26 & 20 \end{pmatrix}$$

Order: $(2\times 3)$ $(3\times 2)$ $(2\times 2)$

**equal**

gives the order of the product

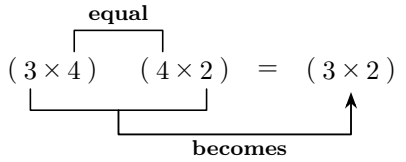
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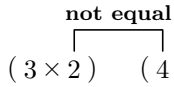
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Matrix multiplication is performed by
"multiplying the **rows of the 1st matrix** on to the **columns of the 2nd matrix**".

Matrix multiplication is possible **only** when the number of columns in the 1st matrix is equal to the number of rows in the 2nd matrix. The order of the resulting matrix product is given by the number of rows in the 1st matrix and the number of columns in the 2nd matrix.




multiplication is possible;
the product is a $3 \times 2$ matrix




multiplication is **not** possible

$$(m \times n \text{ matrix})\,(n \times p \text{ matrix}) = (m \times p \text{ matrix})$$

*(An alternative method, for those who have difficulty using the normal matrix multiplication method.)*

When multiplying $\begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix}$, draw a grid. Write the columns of the 2nd matrix across the top and the rows of the 1st matrix down the side. Each cell is then the row on its left multiplied on to the column above it.

| | 1<br>3<br>4 | 6<br>2<br>0 |
|---|---|---|
| 2 4 1 | $2 \times 1 + 4 \times 3$<br>$+ 1 \times 4 = 18$ | $2 \times 6 + 4 \times 2$<br>$+ 1 \times 0 = 20$ |
| 3 1 5 | $3 \times 1 + 1 \times 3$<br>$+ 5 \times 4 = 26$ | $3 \times 6 + 1 \times 2$<br>$+ 5 \times 0 = 20$ |

$$\therefore \quad \begin{pmatrix} 2 & 4 & 1 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 3 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 18 & 20 \\ 26 & 20 \end{pmatrix}$$

**Note.** In general, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2$ is **not** equal to $\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$! Squaring a matrix means multiplying it by itself, row on to column.

1. Matrix multiplication is **not commutative** in general, i.e. $\mathbf{AB} \neq \mathbf{BA}$. For example, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ but $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Sometimes **AB** exists but **BA** does not exist at all.
2. Matrix multiplication is **associative**, i.e. $\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$; but in general $\mathbf{ABC} \neq \mathbf{ACB}$.
3. Matrix multiplication is **distributive over addition**, i.e. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

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1. **Set up the matrices.** In a word problem, keep a table's rows as rows and its columns as columns, unless the question asks for a different arrangement. A list with one number for each item (a price, a quantity) is written as a **column matrix after** a matrix whose columns are those items, or as a **row matrix in front of** a matrix whose rows are those items.
2. **Match the labels.** Each row and each column of a matrix stands for something (a person, an item, a month): its **label**. Before multiplying $\mathbf{A}$ by $\mathbf{B}$, check that the columns of $\mathbf{A}$ stand for exactly what the rows of $\mathbf{B}$ stand for, one for one and in the same sequence; sizes that fit are not enough. Each row of $\mathbf{AB}$ then belongs to a row of $\mathbf{A}$, and each column to a column of $\mathbf{B}$.
3. **Work it out.** Multiply row on to column; add or subtract only matrices with the same labels (e.g. selling price − cost price = profit). When three matrices are multiplied, set each one up by steps 1 and 2; then either neighbouring pair may be multiplied first, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$, as long as no two matrices swap places.
4. **State what the answer represents**: what each element is, its unit, and which person or item it belongs to ("respectively"). A $1 \times 1$ product has one element: state that number with its unit.

**Useful set-ups**

- **Totals and means.** A row of 1s in front, $\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}\mathbf{M}$, with one 1 for each row of $\mathbf{M}$, gives the total of each column; a column of 1s after $\mathbf{M}$, with one 1 for each column, gives the total of each row. A total over $n$ things, multiplied by $\frac{1}{n}$, gives their mean.
- **Percentage changes.** An increase of $r\%$ multiplies a value by $(100 + r)\%$, a decrease of $r\%$ by $(100 - r)\%$, written as a decimal. The same change for every item is a scalar multiple $k\mathbf{M}$; a different change for each item is a **diagonal matrix** (each item's factor on the principal diagonal, 0s elsewhere) in front of the column of prices.
- **Unknowns.** An unknown element makes one or more elements of the product into expressions. Set the right one equal to the value it must take, or form an inequality if it must be more (or less) than some value, which may be another element of the same product; then solve. A count must be a whole number: take the least (or greatest) whole number that satisfies the inequality.

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## Exercises

### Section 1: Basic Matrix Definitions and Computation

**1.** Simplify each expression to a single matrix.

(a) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(b) $3\begin{pmatrix} 4 & -1 \end{pmatrix} - 2\begin{pmatrix} 5 & 2 \end{pmatrix}$

(c) $2\begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 6 & -3 \\ 9 & 12 \end{pmatrix}$

**2.** Find the matrix $\mathbf{M}$ in each equation.

(a) $3\mathbf{M} + \begin{pmatrix} 1 & -5 \end{pmatrix} = \begin{pmatrix} 7 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -2 \\ 1 & 6 \end{pmatrix} - \frac{1}{2}\mathbf{M} = 2\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$

(c) $3\mathbf{M} - \begin{pmatrix} 2 & -3 \\ 6 & -1 \end{pmatrix} = 4\mathbf{I}$

**3.** Find the unknowns in each matrix equation.

(a) $2\begin{pmatrix} a \\ 2a \end{pmatrix} - 3\begin{pmatrix} b \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \end{pmatrix}$

(b) $3\begin{pmatrix} x & -2 \\ 1 & 2y \end{pmatrix} + 2\begin{pmatrix} 1 & z \\ w & -y \end{pmatrix} = \begin{pmatrix} 11 & 3w \\ -3 & 8 \end{pmatrix}$

(c) $p\begin{pmatrix} 2 \\ 1 \end{pmatrix} + q\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

### Section 2: Matrix Multiplication

**4.** Evaluate each product, or write "Not possible" if the product does not exist.

(a) $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 4 \\ 5 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 1 & 2 \\ 3 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

**5.** It is given that $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$.

(a) Find $\mathbf{AB}$ and $\mathbf{BA}$.

(b) State whether $\mathbf{AB} = \mathbf{BA}$.

(c) Find $\mathbf{A}^2$, and express it in the form $k\mathbf{I}$.

**6.** It is given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & p \\ 0 & \frac{1}{4} \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & -1 \\ 2 & q \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 - 6\mathbf{A} = k\mathbf{I}$, and state the value of $k$.

(b) Given that $\mathbf{AB}$ is the identity matrix, find the value of $p$.

(c) Given that $\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{C} = 2\begin{pmatrix} 9 & -5 \end{pmatrix}$, find the value of $q$.

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## Section 3: Matrix Word Problems

**7.** A workshop makes chairs, tables and bookshelves. The table shows the hours of labour and the mass of timber needed for each item.

| | **Labour (hours)** | **Timber (kg)** |
|---|---|---|
| Chair | 3 | 6 |
| Table | 6 | 20 |
| Bookshelf | 5 | 14 |

Labour costs \$18 per hour and timber costs \$4 per kilogram. It is given that

$$\mathbf{P} = \begin{pmatrix} 3 & 6 \\ 6 & 20 \\ 5 & 14 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 18 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{M} = \mathbf{PQ}.$$

(a) Evaluate $\mathbf{M}$.

(b) State what the elements of $\mathbf{M}$ represent.

(c) A customer orders 10 chairs, 2 tables and 4 bookshelves, and $\mathbf{R} = \begin{pmatrix} 10 & 2 & 4 \end{pmatrix}$.

(i) Evaluate $\mathbf{RM}$.

(ii) State what your answer represents.

**8.** Irfan, Jia Hui and Marcus took a 35-question quiz. Each correct answer earns 4 marks, each wrong answer loses 1 mark, and a question left blank scores 0. Irfan answered every question. The table shows how each student did.

| | **Correct** | **Wrong** | **Blank** |
|---|---|---|---|
| **Irfan** | $x$ | $35 - x$ | 0 |
| **Jia Hui** | 26 | 5 | 4 |
| **Marcus** | 24 | 9 | 2 |

(a) Evaluate $\begin{pmatrix} x & 35-x & 0 \\ 26 & 5 & 4 \\ 24 & 9 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$.

(b) State what your answer to part (a) represents.

(c) Irfan scored more marks than each of the other two students. Use your answer to part (a) to find the least number of questions Irfan answered correctly.

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**9.** A sports shop sells two brands of T-shirt, Breeze and Summit, each in sizes small ($S$), medium ($M$) and large ($L$). The table shows the numbers sold in November and December, and the price of a T-shirt of each size in each month.

| | **November** | | | **December** | | |
|---|---|---|---|---|---|---|
| | $S$ | $M$ | $L$ | $S$ | $M$ | $L$ |
| Breeze | 21 | 27 | 16 | 25 | 33 | 22 |
| Summit | 14 | 23 | 19 | 18 | 29 | 25 |
| **Price of a T-shirt ($)** | 14 | 17 | 21 | 16 | 18 | 24 |

The numbers sold in November are represented by $\mathbf{U} = \begin{pmatrix} 21 & 27 & 16 \\ 14 & 23 & 19 \end{pmatrix}$, and the numbers sold in December by a $2 \times 3$ matrix $\mathbf{V}$.

(a) Write down the matrix $\mathbf{V}$.

(b) Evaluate $\mathbf{A} = \frac{1}{2}(\mathbf{U} + \mathbf{V})$ and state what its elements represent.

(c) The prices in November and in December are represented by

$$\mathbf{X} = \begin{pmatrix} 14 \\ 17 \\ 21 \end{pmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{pmatrix} 16 \\ 18 \\ 24 \end{pmatrix}$$

respectively. Evaluate $\mathbf{UX} + \mathbf{VY}$ and state what its elements represent.

(d) Given that $\mathbf{K} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, evaluate $\mathbf{Z} = (\mathbf{U} + \mathbf{V})\mathbf{K}$ and state what its elements represent.

## Supplementary Word Problems

**10.** Julie runs yoga courses on weekdays and at weekends, at beginner and advanced levels. Each course lasts 6 weeks, with one lesson a week. The matrix $\mathbf{P}$ shows the number of students in each of her current courses.

$$\mathbf{P} = \begin{array}{c} \begin{matrix} \text{Beginner} & \text{Advanced} \end{matrix} \\ \begin{pmatrix} 11 & 6 \\ 7 & 4 \end{pmatrix} \end{array} \begin{matrix} \text{Weekday} \\ \text{Weekend} \end{matrix}$$

(a) Evaluate the matrix $\mathbf{Q} = 6\mathbf{P}$.

(b) Julie charges each student \$25 for a beginner lesson and \$40 for an advanced lesson. Represent these charges as a $2 \times 1$ column matrix $\mathbf{R}$.

(c) Evaluate the matrix $\mathbf{S} = \mathbf{QR}$.

(d) State what the elements of $\mathbf{S}$ represent.

(e) For the next intake, Julie reduces the charge for every lesson by 8%. On weekdays, 12 students sign up for the beginner course and 7 for the advanced course. At weekends, 9 students sign up for the beginner course and 5 for the advanced course. Each course again lasts 6 weeks. Using matrices, calculate the total amount of money Julie earns from this new intake.

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**11.** A company wins a tender to paint the interior of a row of shophouses. The shophouses are of three types, $A$, $B$ and $C$, and the company uses three types of paint. The table shows the volume of each type of paint, in litres, needed to paint one shophouse of each type.

| Paint \ Shophouse | $A$ | $B$ | $C$ |
|---|---|---|---|
| Undercoat (litres) | 14 | 9 | 11 |
| Gloss (litres) | 5 | 4 | 6 |
| Emulsion (litres) | 28 | 20 | 24 |

There are 15 shophouses of type $A$, 12 of type $B$ and 8 of type $C$.

(a) Write down two matrices whose product gives the total volume of each type of paint needed. Evaluate this product.

(b) Undercoat costs \$1.40 per litre, gloss costs \$2.60 per litre and emulsion costs \$1.80 per litre. Write down a matrix whose product with the matrix found in part (a) gives the total cost of the paint for the shophouses. Evaluate this total cost.

**12.** A drinks factory supplies bottled starfruit juice to five stores. The table shows the number of bottles of each size in one delivery to each store, the number of deliveries each store received over a period of 6 months, and the price at which the factory sells a bottle of each size to the stores.

| Store \ Size | **Bottles per delivery** | | | |
|---|---|---|---|---|
| | **100 ml** | **250 ml** | **500 ml** | **Deliveries in 6 months** |
| Rainbow Minimart | 30 | 40 | 20 | 10 |
| Lucky Store | 0 | 25 | 40 | 15 |
| Penny Mart | 40 | 10 | 30 | 8 |
| All-Night Store | 60 | 30 | 0 | 24 |
| Corner Minimart | 20 | 0 | 50 | 9 |
| **Price per bottle** | \$0.60 | \$1.20 | \$2.10 | |

(a) Write down two matrices whose product gives the volume of starfruit juice in one delivery to each store.

(b) Write down two matrices whose product gives the number of bottles of each size the factory supplied over the 6 months. Evaluate this product.

(c) Using the product found in part (b) and a further matrix, find the total amount of money the factory received for the bottles it delivered over the 6 months.

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# Solutions to Exercises

## Section 1

**1**

(a) $\begin{pmatrix}5\\-2\end{pmatrix}+3\begin{pmatrix}-1\\4\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}+\begin{pmatrix}-3\\12\end{pmatrix}$

$=\begin{pmatrix}5-3\\-2+12\end{pmatrix}=\boxed{\begin{pmatrix}2\\10\end{pmatrix}}$

(b) $3\begin{pmatrix}4 & -1\end{pmatrix}-2\begin{pmatrix}5 & 2\end{pmatrix}$

$=\begin{pmatrix}12 & -3\end{pmatrix}-\begin{pmatrix}10 & 4\end{pmatrix}$

$=\begin{pmatrix}12-10 & -3-4\end{pmatrix}=\boxed{\begin{pmatrix}2 & -7\end{pmatrix}}$

(c) $2\begin{pmatrix}3 & -1\\0 & 5\end{pmatrix}-\frac{1}{3}\begin{pmatrix}6 & -3\\9 & 12\end{pmatrix}$

$=\begin{pmatrix}6 & -2\\0 & 10\end{pmatrix}-\begin{pmatrix}2 & -1\\3 & 4\end{pmatrix}$

$=\begin{pmatrix}6-2 & -2+1\\0-3 & 10-4\end{pmatrix}=\boxed{\begin{pmatrix}4 & -1\\-3 & 6\end{pmatrix}}$

**2**

(a) $3\mathbf{M}=\begin{pmatrix}7 & 4\end{pmatrix}-\begin{pmatrix}1 & -5\end{pmatrix}=\begin{pmatrix}6 & 9\end{pmatrix}$

$\mathbf{M}=\frac{1}{3}\begin{pmatrix}6 & 9\end{pmatrix}=\boxed{\begin{pmatrix}2 & 3\end{pmatrix}}$

(b) $\frac{1}{2}\mathbf{M}=\begin{pmatrix}4 & -2\\1 & 6\end{pmatrix}-2\begin{pmatrix}1 & 0\\3 & -1\end{pmatrix}$

$=\begin{pmatrix}4 & -2\\1 & 6\end{pmatrix}-\begin{pmatrix}2 & 0\\6 & -2\end{pmatrix}=\begin{pmatrix}2 & -2\\-5 & 8\end{pmatrix}$

$\mathbf{M}=2\begin{pmatrix}2 & -2\\-5 & 8\end{pmatrix}=\boxed{\begin{pmatrix}4 & -4\\-10 & 16\end{pmatrix}}$

(c) $3\mathbf{M}=4\mathbf{I}+\begin{pmatrix}2 & -3\\6 & -1\end{pmatrix}=\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}+\begin{pmatrix}2 & -3\\6 & -1\end{pmatrix}$

$=\begin{pmatrix}6 & -3\\6 & 3\end{pmatrix}$

$\mathbf{M}=\boxed{\begin{pmatrix}2 & -1\\2 & 1\end{pmatrix}}$

**3** Compare corresponding elements.

(a) $\begin{pmatrix}2a-3b\\4a+3\end{pmatrix}=\begin{pmatrix}1\\15\end{pmatrix}$

$4a+3=15$, so $\boxed{a=3}$

$2(3)-3b=1,\ \ 3b=5$, so $\boxed{b=1\frac{2}{3}}$

(b) $\begin{pmatrix}3x & -6\\3 & 6y\end{pmatrix}+\begin{pmatrix}2 & 2z\\2w & -2y\end{pmatrix}=\begin{pmatrix}11 & 3w\\-3 & 8\end{pmatrix}$

$\begin{pmatrix}3x+2 & -6+2z\\3+2w & 4y\end{pmatrix}=\begin{pmatrix}11 & 3w\\-3 & 8\end{pmatrix}$

$3x+2=11$, so $\boxed{x=3}$

$3+2w=-3$, so $\boxed{w=-3}$

$-6+2z=3w=-9$, so $\boxed{z=-1\frac{1}{2}}$

$4y=8$, so $\boxed{y=2}$

(c) $\begin{pmatrix}2p-q\\p+3q\end{pmatrix}=\begin{pmatrix}7\\7\end{pmatrix}$

$2p-q=7\quad(1)$

$p+3q=7\quad(2)$

$3\times(1)+(2)$: $\ 7p=28$, so $\boxed{p=4}$

From (1): $q=2(4)-7$, so $\boxed{q=1}$

## Section 2

**4**

(a) $\begin{pmatrix}1 & -2\\3 & 0\end{pmatrix}\begin{pmatrix}4 & 1\\2 & -1\end{pmatrix}$

$=\begin{pmatrix}(1)(4)+(-2)(2) & (1)(1)+(-2)(-1)\\(3)(4)+(0)(2) & (3)(1)+(0)(-1)\end{pmatrix}$

$=\boxed{\begin{pmatrix}0 & 3\\12 & 3\end{pmatrix}}$

(b) $\begin{pmatrix}3 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 4\\5 & 0\end{pmatrix}$

$=\begin{pmatrix}(3)(1)+(0)(-2)+(-1)(5) & (3)(2)+(0)(4)+(-1)(0)\end{pmatrix}$

$=\boxed{\begin{pmatrix}-2 & 6\end{pmatrix}}$

(c) Orders $(2\times2)(3\times2)$: $2\neq3$, so $\boxed{\text{Not possible}}$

(d) $\begin{pmatrix}2 & 0 & 1\\-1 & 3 & 2\\4 & 1 & 0\end{pmatrix}\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}(2)(3)+(0)(-1)+(1)(2)\\(-1)(3)+(3)(-1)+(2)(2)\\(4)(3)+(1)(-1)+(0)(2)\end{pmatrix}=\boxed{\begin{pmatrix}8\\-2\\11\end{pmatrix}}$

**5**

(a) $\mathbf{AB}=\begin{pmatrix}1 & 3\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\1 & 4\end{pmatrix}$

$=\begin{pmatrix}(1)(2)+(3)(1) & (1)(0)+(3)(4)\\(2)(2)+(-1)(1) & (2)(0)+(-1)(4)\end{pmatrix}$

$=\boxed{\begin{pmatrix}5 & 12\\3 & -4\end{pmatrix}}$

$\mathbf{BA}=\begin{pmatrix}2 & 0\\1 & 4\end{pmatrix}\begin{pmatrix}1 & 3\\2 & -1\end{pmatrix}$

$=\begin{pmatrix}(2)(1)+(0)(2) & (2)(3)+(0)(-1)\\(1)(1)+(4)(2) & (1)(3)+(4)(-1)\end{pmatrix}$

$=\boxed{\begin{pmatrix}2 & 6\\9 & -1\end{pmatrix}}$

(b) $\boxed{\mathbf{AB}\neq\mathbf{BA}}$

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(c) $\mathbf{A}^2 = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} (1)(1)+(3)(2) & (1)(3)+(3)(-1) \\ (2)(1)+(-1)(2) & (2)(3)+(-1)(-1) \end{pmatrix}$

$= \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \boxed{7\mathbf{I}}$

**6**

(a) $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} (2)(2)+(3)(0) & (2)(3)+(3)(4) \\ (0)(2)+(4)(0) & (0)(3)+(4)(4) \end{pmatrix} = \begin{pmatrix} 4 & 18 \\ 0 & 16 \end{pmatrix}$

$\mathbf{A}^2 - 6\mathbf{A} = \begin{pmatrix} 4 & 18 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} 12 & 18 \\ 0 & 24 \end{pmatrix}$

$= \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} = -8\mathbf{I}$

$\boxed{k = -8}$

(b) $\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & p \\ 0 & \frac{1}{4} \end{pmatrix}$

$= \begin{pmatrix} (2)(\frac{1}{2})+(3)(0) & (2)(p)+(3)(\frac{1}{4}) \\ (0)(\frac{1}{2})+(4)(0) & (0)(p)+(4)(\frac{1}{4}) \end{pmatrix}$

$= \begin{pmatrix} 1 & 2p+\frac{3}{4} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$2p + \frac{3}{4} = 0$, so $\boxed{p = -\frac{3}{8}}$

(c) $\begin{pmatrix} (4)(5)+(-1)(2) & (4)(-1)+(-1)(q) \end{pmatrix} = \begin{pmatrix} 18 & -10 \end{pmatrix}$

$\begin{pmatrix} 18 & -4-q \end{pmatrix} = \begin{pmatrix} 18 & -10 \end{pmatrix}$

$-4 - q = -10$, so $\boxed{q = 6}$

## Section 3

**7**

(a) $\mathbf{M} = \begin{pmatrix} 3 & 6 \\ 6 & 20 \\ 5 & 14 \end{pmatrix}\begin{pmatrix} 18 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} 78 \\ 188 \\ 146 \end{pmatrix}}$

(b) The cost, in dollars, of making a chair, a table and a bookshelf respectively.

(c)(i) $\mathbf{RM} = \begin{pmatrix} 10 & 2 & 4 \end{pmatrix}\begin{pmatrix} 78 \\ 188 \\ 146 \end{pmatrix} = \boxed{\begin{pmatrix} (1740) \end{pmatrix}}$

(ii) The total cost, \$1740, of making 10 chairs, 2 tables and 4 bookshelves.

**8**

(a) $\begin{pmatrix} x & 35-x & 0 \\ 26 & 5 & 4 \\ 24 & 9 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} (x)(4)+(35-x)(-1)+(0)(0) \\ (26)(4)+(5)(-1)+(4)(0) \\ (24)(4)+(9)(-1)+(2)(0) \end{pmatrix}$

$= \boxed{\begin{pmatrix} 5x-35 \\ 99 \\ 87 \end{pmatrix}}$

(b) The marks scored by Irfan, Jia Hui and Marcus respectively in the quiz.

(c) Irfan beats both, so Irfan's score is more than 99.

$5x - 35 > 99$

$5x > 134$

$x > 26.8$

Least number of correct answers $= \boxed{27}$

**9**

(a) $\mathbf{V} = \boxed{\begin{pmatrix} 25 & 33 & 22 \\ 18 & 29 & 25 \end{pmatrix}}$

(b) $\mathbf{A} = \frac{1}{2}\begin{pmatrix} 21+25 & 27+33 & 16+22 \\ 14+18 & 23+29 & 19+25 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 46 & 60 & 38 \\ 32 & 52 & 44 \end{pmatrix} = \boxed{\begin{pmatrix} 23 & 30 & 19 \\ 16 & 26 & 22 \end{pmatrix}}$

The mean number of T-shirts sold per month, over November and December, in sizes $S$, $M$ and $L$ (columns) of the Breeze and Summit brands (rows).

(c) $\mathbf{UX} = \begin{pmatrix} 21 & 27 & 16 \\ 14 & 23 & 19 \end{pmatrix}\begin{pmatrix} 14 \\ 17 \\ 21 \end{pmatrix} = \begin{pmatrix} 1089 \\ 986 \end{pmatrix}$

$\mathbf{VY} = \begin{pmatrix} 25 & 33 & 22 \\ 18 & 29 & 25 \end{pmatrix}\begin{pmatrix} 16 \\ 18 \\ 24 \end{pmatrix} = \begin{pmatrix} 1522 \\ 1410 \end{pmatrix}$

$\mathbf{UX} + \mathbf{VY} = \begin{pmatrix} 1089+1522 \\ 986+1410 \end{pmatrix} = \boxed{\begin{pmatrix} 2611 \\ 2396 \end{pmatrix}}$

The total amount of money, in dollars, taken from the sale of Breeze and of Summit T-shirts respectively over November and December.

(d) $\mathbf{Z} = \begin{pmatrix} 46 & 60 & 38 \\ 32 & 52 & 44 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 144 \\ 128 \end{pmatrix}}$

The total numbers of Breeze and of Summit T-shirts respectively sold over November and December.

## Supplementary Word Problems

**10**

(a) $\mathbf{Q} = 6\begin{pmatrix} 11 & 6 \\ 7 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 66 & 36 \\ 42 & 24 \end{pmatrix}}$

[*the lessons attended in each course over the 6 weeks*]

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(b) $\mathbf{R} = \boxed{\begin{pmatrix} 25 \\ 40 \end{pmatrix}}$

(c) $\mathbf{S} = \begin{pmatrix} 66 & 36 \\ 42 & 24 \end{pmatrix} \begin{pmatrix} 25 \\ 40 \end{pmatrix} = \boxed{\begin{pmatrix} 3090 \\ 2010 \end{pmatrix}}$

(d) The amount of money, in dollars, Julie earns from her weekday courses and from her weekend courses respectively over the 6 weeks.

(e) New charges: $0.92 \begin{pmatrix} 25 \\ 40 \end{pmatrix} = \begin{pmatrix} 23.00 \\ 36.80 \end{pmatrix}$

$[100\% - 8\% = 92\%]$

Earnings from the new intake, in dollars:

$6 \begin{pmatrix} 12 & 7 \\ 9 & 5 \end{pmatrix} \begin{pmatrix} 23.00 \\ 36.80 \end{pmatrix}$

$= \begin{pmatrix} 72 & 42 \\ 54 & 30 \end{pmatrix} \begin{pmatrix} 23.00 \\ 36.80 \end{pmatrix} = \begin{pmatrix} 3201.60 \\ 2346.00 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3201.60 \\ 2346.00 \end{pmatrix} = \begin{pmatrix} 5547.60 \end{pmatrix}$

Total = $\boxed{\$5547.60}$

**11**

(a) $\begin{pmatrix} 14 & 9 & 11 \\ 5 & 4 & 6 \\ 28 & 20 & 24 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 8 \end{pmatrix} = \boxed{\begin{pmatrix} 406 \\ 171 \\ 852 \end{pmatrix}}$

[*rows: undercoat, gloss, emulsion; columns of the first matrix: types A, B, C*]

The company needs 406 litres of undercoat, 171 litres of gloss and 852 litres of emulsion.

(b) $\begin{pmatrix} 1.40 & 2.60 & 1.80 \end{pmatrix}$ [*written in front of the product*]

$\begin{pmatrix} 1.40 & 2.60 & 1.80 \end{pmatrix} \begin{pmatrix} 406 \\ 171 \\ 852 \end{pmatrix} = \begin{pmatrix} 2546.60 \end{pmatrix}$

Total cost = $\boxed{\$2546.60}$

**12**

(a) $\begin{pmatrix} 30 & 40 & 20 \\ 0 & 25 & 40 \\ 40 & 10 & 30 \\ 60 & 30 & 0 \\ 20 & 0 & 50 \end{pmatrix} \begin{pmatrix} 100 \\ 250 \\ 500 \end{pmatrix} = \boxed{\begin{pmatrix} 23\,000 \\ 26\,250 \\ 21\,500 \\ 13\,500 \\ 27\,000 \end{pmatrix}}$

The volumes, in ml, in one delivery to the five stores respectively (23, 26.25, 21.5, 13.5 and 27 litres).

(b) $\begin{pmatrix} 10 & 15 & 8 & 24 & 9 \end{pmatrix} \begin{pmatrix} 30 & 40 & 20 \\ 0 & 25 & 40 \\ 40 & 10 & 30 \\ 60 & 30 & 0 \\ 20 & 0 & 50 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 2240 & 1575 & 1490 \end{pmatrix}}$

2240 bottles of 100 ml, 1575 of 250 ml and 1490 of 500 ml.

(c) $\begin{pmatrix} 2240 & 1575 & 1490 \end{pmatrix} \begin{pmatrix} 0.60 \\ 1.20 \\ 2.10 \end{pmatrix} = \begin{pmatrix} 6363 \end{pmatrix}$

Total = $\boxed{\$6363}$

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