

Differentiation of Trigonometric, Logarithmic and Exponential Functions

Derivatives of Trigonometric, Logarithmic and Exponential Functions, and Their Applications
Secondary 4 — G3 Additional Mathematics

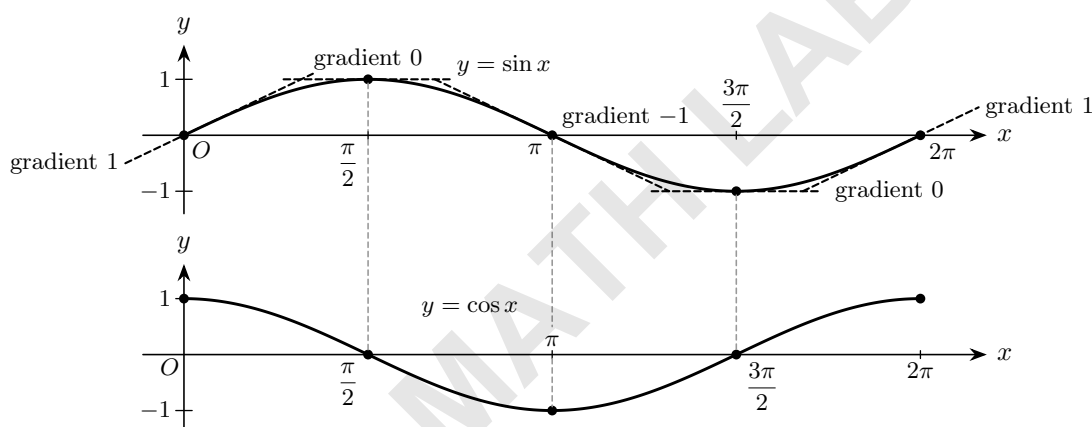
Notes

Derivatives of the trigonometric functions

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \frac{d}{dx}[\cos(x)] = -\sin(x) \quad \frac{d}{dx}[\tan(x)] = \sec^2(x)$$

Trigo differentiation: the angle must be in radians.

The gradient of $y = \sin x$ at any point is the value of $\cos x$ there. Where the sine curve climbs most steeply (at $x = 0$ and $x = 2\pi$) its gradient is 1, at its peaks and troughs the gradient is 0, and where it falls most steeply (at $x = \pi$) the gradient is -1 : exactly the values of $\cos x$ below.



With a function of x inside: the chain rule

$$\frac{d}{dx}[\sin(f(x))] = f'(x) \times \cos(f(x))$$

$$\frac{d}{dx}[\cos(f(x))] = f'(x) \times [-\sin(f(x))]$$

$$\frac{d}{dx}[\tan(f(x))] = f'(x) \times \sec^2(f(x))$$

Trigo differentiation: the angle must be in radians.

Read $\sin(f(x))$ as two steps: an *inner* function $u = f(x)$ fed into an *outer* function $\sin u$. Differentiate the outer function, keeping the inner function unchanged inside it, then multiply by the derivative of the inner function.

$$\begin{array}{c}
 x \longrightarrow \boxed{\text{inner: } u = x^2 + 1} \xrightarrow{u} \boxed{\text{outer: } y = \sin u} \longrightarrow y \\
 \frac{du}{dx} = 2x \qquad \qquad \frac{dy}{du} = \cos u \\
 \\
 \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos(x^2 + 1) \times 2x = 2x \cos(x^2 + 1)
 \end{array}$$

Powers of the trigonometric functions

$$\frac{d}{dx}(\sin^n x) = \frac{d}{dx}((\sin x)^n) = n(\sin x)^{n-1}(\cos x)$$

$$\frac{d}{dx}(\cos^n x) = \frac{d}{dx}((\cos x)^n) = n(\cos x)^{n-1}(-\sin x)$$

$$\frac{d}{dx}(\tan^n x) = \frac{d}{dx}((\tan x)^n) = n(\tan x)^{n-1}(\sec^2 x)$$

$\sin^n x$ means $(\sin x)^n$: the power is the *outermost* function. When the angle is itself a function of x , there are three layers, and the chain rule multiplies one derivative for each layer, working from the outside in.

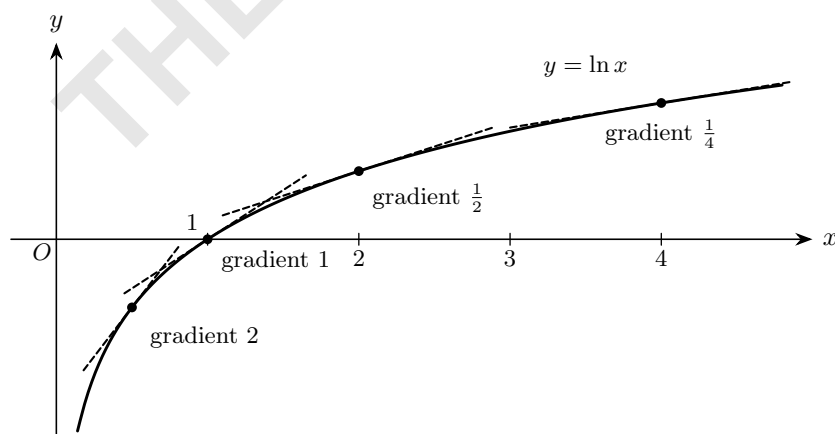
$$\begin{array}{ccccccc}
 x & \longrightarrow & \boxed{u = 4x} & \xrightarrow{u} & \boxed{v = \cos u} & \xrightarrow{v} & \boxed{y = v^3} \longrightarrow y \\
 & & \frac{du}{dx} = 4 & & \frac{dv}{du} = -\sin u & & \frac{dy}{dv} = 3v^2
 \end{array}$$

$$\frac{d}{dx}(\cos^3 4x) = \underbrace{3(\cos 4x)^2}_{\text{power}} \times \underbrace{(-\sin 4x)}_{\text{cos}} \times \underbrace{4}_{\text{angle}} = -12 \cos^2 4x \sin 4x$$

Derivatives of the natural logarithm

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

The tangents to $y = \ln x$ flatten as x increases: the gradient at x is $\frac{1}{x}$, so it is 2 at $x = \frac{1}{2}$, 1 at $x = 1$, $\frac{1}{2}$ at $x = 2$ and $\frac{1}{4}$ at $x = 4$. The gradient is always positive, so $\ln x$ is an increasing function, but it increases ever more slowly. For $\ln(f(x))$, the chain rule gives $\frac{1}{f(x)} \times f'(x)$: the derivative of the inside, divided by the inside.

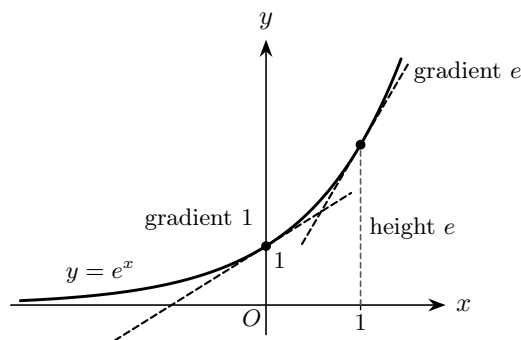


Derivatives of the exponential function

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}[e^{f(x)}] = f'(x) \times e^{f(x)}$$

$y = e^x$ is the function that is its own derivative: at every point of the curve, the **gradient equals the height**. At $(0, 1)$ the tangent has gradient 1; at $(1, e)$ it has gradient $e \approx 2.72$. The curve never meets the x -axis, so $e^x > 0$ for every x , and the gradient is always positive.

For $e^{f(x)}$, keep the power unchanged and multiply by the derivative of the power, $f'(x)$.



The second derivative

$\frac{d^2y}{dx^2}$ is the derivative of $\frac{dy}{dx}$: differentiate y , simplify the result, then differentiate it again. In function notation, $f''(x)$ is the derivative of $f'(x)$.

Equations in y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. To show that such an equation holds, to find the unknown constants in it, or to find the value of an expression in these terms:

- 1) Simplify y first where you can: use the laws of logarithms on $\ln$, and divide each term of a numerator by an exponential denominator.
- 2) Find $\frac{dy}{dx}$, then $\frac{d^2y}{dx^2}$. A product such as $(ax + b)e^{kx}$ or $e^{kx} \sin ax$ needs the product rule; take out the common factor, such as e^{kx} , each time.
- 3) Substitute y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the expression, take the common factor outside a bracket, and collect the like terms inside it.
- 4) To **show** a result, simplify until you reach the given right-hand side, and write (shown). To find the **value** of an expression, simplify it until x no longer appears. To **find constants** in a result that holds for all x , write both sides in the same form and **compare the coefficients** of matching terms.

Trigonometric forms. To write a derivative in a stated form, use $\sin 2A = 2 \sin A \cos A$, $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$, $\sin^2 A + \cos^2 A = 1$ and the addition formulae.

Gradients of the tangent and the normal

At the point (x_1, y_1) on a curve, the gradient of the **tangent** is m , the value of $\frac{dy}{dx}$ at $x = x_1$. The **normal** is perpendicular to the tangent, so the product of their gradients is -1 and the gradient of the normal is $-\frac{1}{m}$. (If $m = 0$, the normal is the vertical line $x = x_1$.)

- 1) Find $\frac{dy}{dx}$.
- 2) Find both coordinates of the point: substitute x into the equation of the curve. If only y is given, solve for x (for a curve $y = e^{f(x)}$, take $\ln$ of both sides of $e^{f(x)} = k$).
- 3) Substitute x into $\frac{dy}{dx}$ to get the tangent's gradient m ; the normal's gradient is $-1 \div m$.
- 4) Write $y = (\text{gradient})x + c$ with the gradient of the line you want, substitute the point to find c , and state the equation. An exact answer keeps e , $\ln$, π and surds.

- A line meets the x -axis where $y = 0$ and the y -axis where $x = 0$. A triangle with one side on an axis has area $\frac{1}{2} \times \text{base} \times \text{height}$ (in units²), the height being the distance of the third vertex from that axis.
- **Parallel to a given line, or a given gradient:** make y the subject of the line's equation to read its gradient; parallel lines have equal gradients. If it is the normal that is parallel, or whose gradient is given, the tangent's gradient is $-1 \div (\text{that gradient})$. Set $\frac{dy}{dx}$ equal to the tangent's gradient and solve; reject a root at which the curve, in its equation as given, is not defined (the whole argument of $\ln$ must be positive) or which lies outside a stated range of x .
- **Through a given point:** if the tangent at $(a, f(a))$ passes through another point (p, q) , the gradient between the two points equals the gradient of the tangent: $\frac{f(a) - q}{a - p} = f'(a)$; solve for a . A normal works the same way (when $f'(a) \neq 0$), with $-\frac{1}{f'(a)}$ in place of $f'(a)$.

Connected rates of change

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \qquad \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dx}{dy}, \text{ where } \frac{dx}{dy} = 1 \div \frac{dy}{dx}$$

- 1) Write down the given rate with its **sign**: a quantity that is increasing has a positive rate, and one that is decreasing has a negative rate.
- 2) Differentiate the equation that connects the variables, and substitute the value at the given instant to get $\frac{dy}{dx}$ as a number.
- 3) Link the given rate and $\frac{dy}{dx}$ by the chain rule above, and state the units of the rate found.
- 4) A negative answer means that the quantity is decreasing at that instant.

Lengths and angles in a figure. First express the quantity in terms of the angle θ , using trigonometric ratios, the sine rule, the area $\frac{1}{2}ab \sin C$ or the sector area $\frac{1}{2}r^2\theta$. Differentiate with respect to θ (in radians), then link the rates: $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$. A longer chain, such as $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$, works in the same way. A rate asked for in terms of θ is left as an expression in θ .

Note: e^{-x} is always positive and can never be equal to zero. Hence it can be divided away from an inequality in the same way as a positive constant, as when finding where a function is increasing or decreasing.

Always simplify $\ln$ before differentiating.

Stationary points

A **stationary point** is a point where $\frac{dy}{dx} = 0$, and its **stationary value** is the value of y there. A maximum or a minimum point is also called a **turning point**.

- 1) In a practical problem, first express the quantity in terms of one variable, often an angle θ , from the figure. Simplify y where you can, find $\frac{dy}{dx}$, and factorise it where possible.
- 2) Solve $\frac{dy}{dx} = 0$. An exponential factor gives no root, since $e^{f(x)} > 0$. Solve $\ln x = k$ as $x = e^k$, and $e^{f(x)} = k$ as $f(x) = \ln k$ when $k > 0$ (there is no solution when $k \leq 0$). Write a trigonometric derivative in one ratio first (using $\sin^2 \theta + \cos^2 \theta = 1$ or a double angle formula, or dividing by $\cos \theta$ to obtain $\tan \theta$); solve a quadratic in $\sin \theta$ or $\cos \theta$ by factorising or by the formula, and reject a value outside -1 to 1 , or one that gives no angle in the stated range.
- 3) Find y by substituting each x -coordinate into the equation of the curve.

4) Determine the **nature** of each point. **Second derivative test:** substitute x into $\frac{d^2y}{dx^2}$. A positive value gives a **minimum**, a negative value a **maximum**; if $\frac{d^2y}{dx^2}$ has one sign for every x in the domain, saying so is enough. If it is 0, the test is inconclusive: use the first derivative test.

First derivative test: find the sign of $\frac{dy}{dx}$ for x slightly less than and slightly greater than the x -coordinate of the point. The signs $-$, 0 , $+$ give a minimum; $+$, 0 , $-$ a maximum; no change of sign, a point of inflexion.

- **Trigonometric equations:** work out the range of the whole angle (such as $2x$) from the range of x first, use the basic angle and the quadrants that the sign allows, and find x last. Angles are in radians.
- **No stationary point:** show that $\frac{dy}{dx} = 0$ has no solution in the domain (for instance, $\frac{dy}{dx} > 0$ throughout).
- An exact answer (in e , $\ln$, π or surds) may be left exact, and must be when the question asks for it; give an answer that is not exact to 3 significant figures, keeping more figures in the working.

THE MATH LAB

Exercises

Section 1: Differentiation of Trigonometric Functions

1. Find the derivative of each expression with respect to x .

- (a) $4 \sin x - 2 \tan x$ (b) $5x^3 - 3 \cos x$ (c) $6 \tan x + 7$ (d) $2x^4 + 3 \sin x$
(e) $2 \sin 5x - \tan 3x$ (f) $3 \cos 4x + 2x$ (g) $\sin 6x - \cos 2x$ (h) $\tan \left(3x - \frac{\pi}{6}\right)$
(i) $\cos \left(\frac{x}{\sqrt{3}} + \frac{\pi}{4}\right)$ (j) $\sin \left(\frac{\pi}{6} - \frac{5}{2}x\right)$

2. Obtain $\frac{dy}{dx}$ when y is each of the following.

- (a) $\cos 2\pi x - \sin 5x$ (b) $3 - 2 \cos 5x$ (c) $\sin \left(2x - \frac{\pi}{3}\right)$
(d) $4 \tan \frac{\pi x}{2}$ (e) $\sqrt{3 - \cos 4x}$ (f) $\frac{x}{\sin 3x}$

3. Differentiate with respect to x , using the product, quotient or chain rule as appropriate.

- (a) $(2 - x^2) \tan 2x$ (b) $x^3 \sin x$ (c) $5x^2 \cos 2x$
(d) $\frac{3 + \cos x}{\sin x}$ (e) $\frac{\sin 2x}{\cos 5x}$ (f) $\sin(2x^3 + 1)$

Section 2: Derivatives of $\sin^n x$, $\cos^n x$ and $\tan^n x$

4. Find the derivative of each of the following with respect to x .

- (a) $5 \cos^3 x$ (b) $2 \sin^4 x + 3 \cos 2x$ (c) $4x^2 - 3 \tan^4 x$
(d) $\sin^2(3x - 2)$ (e) $6 \cos^4(2x + \pi)$ (f) $3 \tan^2(4x + 1)$

5. Differentiate each expression with respect to x .

- (a) $3x^3 \sin^4(2 - x)$ (b) $\cos^3(\pi - 2x)$ (c) $x^2 \sin^3 x$
(d) $\frac{2 + 3 \cos^2 x}{x}$ (e) $\frac{1}{\cos 5x}$

6. Find the derivative of each of the following with respect to x .

- (a) $3x^4 \sin x$ (b) $\operatorname{cosec} 4x$ (c) $\cot 2x$
(d) $\frac{4 \cos^2 2x}{x}$ (e) $\frac{1}{\sqrt{2 + \sin^2 3x}}$

Section 3: Derivatives of Logarithmic Functions

7. Find the derivative of each expression with respect to x .

- (a) $\ln(7 - 2x)$ (b) $\ln(x^3 + 5)$ (c) $\ln(x^2 - 9)$ (d) $\ln[(3x + 2)^4]$
(e) $\ln[(1 - 4x)^2]$ (f) $\ln \sqrt[3]{2x + 3}$ (g) $\ln \frac{5}{x^2}$ (h) $\ln \frac{3}{4x + 1}$

8. Differentiate each of the following with respect to x .

- (a) $\ln \sin 4x$ (b) $\ln(1 + \tan x)$ (c) $\ln \sqrt{\frac{2+x}{1-2x}}$ (d) $\ln \sqrt[3]{\frac{x}{(x+3)^2}}$
(e) $2x^3 \ln 3x$ (f) $(x^2 - 1)^2 \ln x$ (g) $\frac{\ln x}{x^3}$ (h) $\ln \left[(1 + \sqrt{x})^3 \right]$

9. Find $\frac{dy}{dx}$ in each case.

- (a) $y = \log_2 5x$ (b) $y = \lg(3x - 1)$ (c) $y = \log_b(2x^2 - 3)$
(d) $e^y = x^4 + 3x^2$ (e) $e^y = \sin 3x$ (f) $e^y = \sqrt{1 + x^2}$

Section 4: Derivatives of Exponential Functions

10. Differentiate each of the following with respect to x .

- (a) e^{5-3x} (b) e^{2x^2-x} (c) $4e^{1+2x} - 3x^2$ (d) $\frac{5}{e^{2x}}$
(e) $e^{3x} - \frac{1}{e^{3x}}$ (f) $\frac{1}{3}e^{\frac{3}{2}x} + 4e^{-\frac{1}{2}x}$ (g) $e^{(x+1)^2}$ (h) $\frac{1}{e^{x^2}}$

11. Find $\frac{dy}{dx}$ for each of the following.

- (a) $y = e^x \sqrt{x}$ (b) $y = e^{-\frac{4}{x}}$ (c) $y = \frac{1}{\sqrt[3]{e^x}}$
(d) $y = (e^x + e^{-x})^2$ (e) $y = \frac{e^{4x} + 3e^x}{e^{2x}}$ (f) $y = \frac{e^x + 2}{e^{2x}}$

12. Find the derivative of each expression with respect to x .

- (a) $(2 - e^{-x})^4$ (b) $e^{1-x} \cos 3x$ (c) $\sqrt{1 + e^{2x}}$
(d) $e^{2+\cos 3x}$ (e) $\frac{e^{2x}}{\sin x}$

Section 5: Simple Applications of Trigonometric, Logarithmic and Exponential Functions

13. The curve $y = pe^{2x} + qe^{-x}$, where p and q are constants, satisfies $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6e^{2x} + 4e^{-x}$ for all values of x . Find the value of p and of q .

14. The variables x and y are related by $y = 2xe^{-3x}$.

- (i) Find $\frac{dy}{dx}$.
(ii) Given that $q = e^{3x} \left(\frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + 6y \right)$, find the value of q .

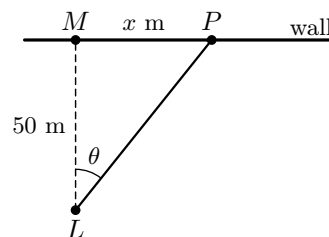
15. Show that $\frac{d}{dx} \left[5 \sin^2 \left(x + \frac{\pi}{4} \right) \right]$ can be written as $k \cos 2x$, and state the value of the constant k .

Section 6: Tangents and Normals

16. A curve has the equation $y = \sin x + \ln x$.
- Find $\frac{dy}{dx}$.
 - Find the equation of the tangent to the curve at the point where $x = \frac{\pi}{2}$.
17. The point P lies on the curve $y = e^{2x}$ and has y -coordinate 4.
- Show that the x -coordinate of P is $\ln 2$.
 - Find the equation of the tangent to the curve at P .
 - Find the equation of the normal to the curve at P .
 - The tangent and the normal at P meet the y -axis at A and B respectively. Find the exact area of triangle APB .
18. A curve has the equation $y = \ln\left(\frac{x}{x+2}\right)$. Find, in exact form, the coordinates of the points on the curve at which the normal is parallel to the line $2y + 8x = 5$.

Section 7: Rates of Change

19. The equation of a curve is $y = \ln \sqrt[3]{3x^2 + 1}$.
- Show that $\frac{dy}{dx} = \frac{kx}{3x^2 + 1}$, stating the value of the constant k .
 - A point moves along the curve so that its y -coordinate decreases at a constant rate of 3 units per second. Find the rate of change of its x -coordinate at the instant when $x = -1$.
20. The variables x and y are connected by $y = x \tan 2x$, where $0 < x < \frac{\pi}{4}$. Given that y increases at a constant rate of 2 units per second, find the exact rate of change of x at the instant when $x = \frac{\pi}{8}$.
21. A searchlight at L is 50 m from a long straight wall, and M is the point on the wall nearest to L . The beam turns so that the angle θ between the beam and LM increases at a constant rate of 0.1 radian per second. The beam lights up the point P on the wall, where $MP = x$ m.
- Express x in terms of θ .
 - Find the speed at which the lit point P moves along the wall at the instant when $\theta = \frac{\pi}{3}$.

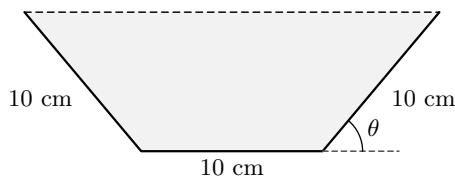


Section 8: Increasing and Decreasing Functions

22. Find the range of values of x for which $y = (2x - 1)e^{-x^2}$ is an increasing function.
23. It is given that $y = \ln \sqrt{\frac{2x-1}{2x+1}}$, where $x > \frac{1}{2}$. Prove that y is an increasing function.
24. A curve has the equation $y = \ln(5 - 2x)$.
- Show that y is a decreasing function.
 - Find the equation of the normal to the curve at the point where it meets the y -axis.
 - Sketch the curve $y = \ln(5 - 2x)$.

Section 9: Stationary Points, Maxima and Minima

25. Find the coordinates of the stationary points of the curve $y = x^2 e^{-x}$, and determine the nature of each.
26. Find the exact coordinates of the stationary point of the curve $y = 2x \ln x - 5x$, and determine its nature.
27. The cross-section of a water trough is an isosceles trapezium. Its base is 10 cm wide and each sloping side is 10 cm long. Each sloping side makes an angle of θ radians with the horizontal, as shown, where $0 < \theta < \frac{\pi}{2}$. The area of the cross-section is $A \text{ cm}^2$.



- (a) Show that $A = 100 \sin \theta (1 + \cos \theta)$.
- (b) Find the value of θ for which A has a stationary value.
- (c) Find this value of A , and determine whether it is a maximum or a minimum.

Solutions to Exercises

Section 1 — Differentiation of trigonometric functions (Questions 1–3)

- 1 (a) $4 \cos x - 2 \sec^2 x$
 (b) $15x^2 + 3 \sin x$
 (c) $6 \sec^2 x$
 (d) $8x^3 + 3 \cos x$
 (e) $2(5) \cos 5x - 3 \sec^2 3x = 10 \cos 5x - 3 \sec^2 3x$
 (f) $3(4)(-\sin 4x) + 2 = -12 \sin 4x + 2$
 (g) $6 \cos 6x + 2 \sin 2x$
 (h) $3 \sec^2 \left(3x - \frac{\pi}{6} \right)$
 (i) $-\frac{1}{\sqrt{3}} \sin \left(\frac{x}{\sqrt{3}} + \frac{\pi}{4} \right)$
 (j) $\left(-\frac{5}{2} \right) \cos \left(\frac{\pi}{6} - \frac{5}{2}x \right) = -\frac{5}{2} \cos \left(\frac{\pi}{6} - \frac{5}{2}x \right)$

- 2 (a) $-2\pi \sin 2\pi x - 5 \cos 5x$
 (b) $-2(5)(-\sin 5x) = 10 \sin 5x$
 (c) $2 \cos \left(2x - \frac{\pi}{3} \right)$
 (d) $4 \left(\frac{\pi}{2} \right) \sec^2 \frac{\pi x}{2} = 2\pi \sec^2 \frac{\pi x}{2}$
 (e) $\frac{d}{dx} (3 - \cos 4x)^{\frac{1}{2}} = \frac{1}{2} (3 - \cos 4x)^{-\frac{1}{2}} (4 \sin 4x)$
 $= \frac{2 \sin 4x}{\sqrt{3 - \cos 4x}}$
 (f) $\frac{\sin 3x(1) - x(3 \cos 3x)}{\sin^2 3x} = \frac{\sin 3x - 3x \cos 3x}{\sin^2 3x}$

- 3 (a) Product rule: $-2x \tan 2x + (2 - x^2) (2 \sec^2 2x)$
 $= -2x \tan 2x + 2(2 - x^2) \sec^2 2x$
 (b) $3x^2 \sin x + x^3 \cos x = x^2(3 \sin x + x \cos x)$
 (c) $10x \cos 2x + 5x^2(-2 \sin 2x)$
 $= 10x(\cos 2x - x \sin 2x)$
 (d) $\frac{\sin x(-\sin x) - (3 + \cos x)(\cos x)}{\sin^2 x}$
 $= \frac{-\sin^2 x - 3 \cos x - \cos^2 x}{\sin^2 x} = -\frac{1 + 3 \cos x}{\sin^2 x}$
 (e) $\frac{\cos 5x(2 \cos 2x) - \sin 2x(-5 \sin 5x)}{\cos^2 5x}$
 $= \frac{2 \cos 2x \cos 5x + 5 \sin 2x \sin 5x}{\cos^2 5x}$

$$(f) \cos(2x^3 + 1) \times 6x^2 = 6x^2 \cos(2x^3 + 1)$$

Section 2 — Derivatives of $\sin^n x$, $\cos^n x$ and $\tan^n x$ (Questions 4–6)

- 4 (a) $5(3) \cos^2 x(-\sin x) = -15 \cos^2 x \sin x$
 (b) $2(4) \sin^3 x \cos x + 3(2)(-\sin 2x)$
 $= 8 \sin^3 x \cos x - 6 \sin 2x$
 (c) $8x - 3(4) \tan^3 x \sec^2 x = 8x - 12 \tan^3 x \sec^2 x$
 (d) $2 \sin(3x - 2)(3 \cos(3x - 2))$
 $= 6 \sin(3x - 2) \cos(3x - 2)$
 (e) $6(4) \cos^3(2x + \pi)(-2 \sin(2x + \pi))$
 $= -48 \cos^3(2x + \pi) \sin(2x + \pi)$
 (f) $3(2) \tan(4x + 1)(4 \sec^2(4x + 1))$
 $= 24 \tan(4x + 1) \sec^2(4x + 1)$

5 (a) Product rule:

$$9x^2 \sin^4(2 - x) + 3x^3 \cdot 4 \sin^3(2 - x) \cos(2 - x)(-1)$$

$$= 9x^2 \sin^4(2 - x) - 12x^3 \sin^3(2 - x) \cos(2 - x)$$

$$= 3x^2 \sin^3(2 - x) [3 \sin(2 - x) - 4x \cos(2 - x)]$$

- (b) $3 \cos^2(\pi - 2x)(-\sin(\pi - 2x))(-2)$
 $= 6 \cos^2(\pi - 2x) \sin(\pi - 2x)$
 (c) $2x \sin^3 x + x^2 \cdot 3 \sin^2 x \cos x$
 $= x \sin^2 x (2 \sin x + 3x \cos x)$
 (d) $\frac{x(3 \cdot 2 \cos x(-\sin x)) - (2 + 3 \cos^2 x)(1)}{x^2}$
 $= \frac{-6x \sin x \cos x - 2 - 3 \cos^2 x}{x^2}$
 (e) $\frac{1}{\cos 5x} = (\cos 5x)^{-1}$

$$-(\cos 5x)^{-2}(-5 \sin 5x) = \frac{5 \sin 5x}{\cos^2 5x}$$

$$6 (a) 12x^3 \sin x + 3x^4 \cos x = 3x^3(4 \sin x + x \cos x)$$

$$(b) \operatorname{cosec} 4x = (\sin 4x)^{-1}$$

$$-(\sin 4x)^{-2}(4 \cos 4x) = -\frac{4 \cos 4x}{\sin^2 4x}$$

$$= -4 \cot 4x \operatorname{cosec} 4x$$

$$(c) \cot 2x = \frac{\cos 2x}{\sin 2x}$$

$$\frac{\sin 2x(-2 \sin 2x) - \cos 2x(2 \cos 2x)}{\sin^2 2x}$$

$$= \frac{-2(\sin^2 2x + \cos^2 2x)}{\sin^2 2x} = \frac{-2}{\sin^2 2x}$$

$$= -2 \operatorname{cosec}^2 2x$$

$$\begin{aligned} \text{(d)} \quad & \frac{x \cdot 4(2 \cos 2x)(-2 \sin 2x) - 4 \cos^2 2x}{x^2} \\ &= \frac{-16x \sin 2x \cos 2x - 4 \cos^2 2x}{x^2} \\ &= \boxed{-\frac{4 \cos 2x (4x \sin 2x + \cos 2x)}{x^2}} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad & (2 + \sin^2 3x)^{-\frac{1}{2}} \\ & -\frac{1}{2} (2 + \sin^2 3x)^{-\frac{3}{2}} \cdot 2 \sin 3x \cdot 3 \cos 3x \\ &= \boxed{\frac{-3 \sin 3x \cos 3x}{\sqrt{(2 + \sin^2 3x)^3}}} \end{aligned}$$

Section 3 — Derivatives of logarithmic functions (Questions 7–9)

$$7 \text{ (a)} \quad \frac{-2}{7-2x} = \boxed{\frac{2}{2x-7}}$$

$$\text{(b)} \quad \boxed{\frac{3x^2}{x^3+5}}$$

$$\text{(c)} \quad \boxed{\frac{2x}{x^2-9}}$$

$$\text{(d)} \quad \frac{4(3x+2)^3(3)}{(3x+2)^4} = \boxed{\frac{12}{3x+2}}$$

$$\text{(e)} \quad \frac{2(1-4x)(-4)}{(1-4x)^2} = \frac{-8}{1-4x} = \boxed{\frac{8}{4x-1}}$$

$$\text{(f)} \quad \frac{1}{3} \ln(2x+3); \quad \boxed{\frac{2}{3(2x+3)}}$$

$$\text{(g)} \quad \ln 5 - 2 \ln x; \quad \boxed{-\frac{2}{x}}$$

$$\text{(h)} \quad \ln 3 - \ln(4x+1); \quad \boxed{-\frac{4}{4x+1}}$$

$$8 \text{ (a)} \quad \frac{4 \cos 4x}{\sin 4x} = \boxed{4 \cot 4x}$$

$$\text{(b)} \quad \boxed{\frac{\sec^2 x}{1 + \tan x}}$$

$$\begin{aligned} \text{(c)} \quad & \frac{1}{2} \ln(2+x) - \frac{1}{2} \ln(1-2x) \\ & \frac{1}{2(2+x)} + \frac{2}{2(1-2x)} = \frac{(1-2x) + 2(2+x)}{2(2+x)(1-2x)} \\ &= \boxed{\frac{5}{2(2+x)(1-2x)}} \end{aligned}$$

$$\text{(d)} \quad \frac{1}{3} [\ln x - 2 \ln(x+3)]$$

$$\frac{1}{3} \left[\frac{1}{x} - \frac{2}{x+3} \right] = \frac{1}{3} \cdot \frac{x+3-2x}{x(x+3)} = \boxed{\frac{3-x}{3x(x+3)}}$$

$$\text{(e)} \quad 6x^2 \ln 3x + 2x^3 \left(\frac{3}{3x} \right) = \boxed{2x^2(3 \ln 3x + 1)}$$

$$\begin{aligned} \text{(f)} \quad & 2(x^2-1)(2x) \ln x + (x^2-1)^2 \left(\frac{1}{x} \right) \\ &= \boxed{4x(x^2-1) \ln x + \frac{(x^2-1)^2}{x}} \end{aligned}$$

$$\text{(g)} \quad \frac{x^3 \left(\frac{1}{x} \right) - \ln x (3x^2)}{x^6} = \boxed{\frac{1-3 \ln x}{x^4}}$$

$$\text{(h)} \quad 3 \ln(1+\sqrt{x});$$

$$\frac{3 \left(\frac{1}{2\sqrt{x}} \right)}{1+\sqrt{x}} = \frac{3}{2\sqrt{x}(1+\sqrt{x})} = \boxed{\frac{3}{2(\sqrt{x}+x)}}$$

$$9 \text{ (a)} \quad y = \frac{\ln 5x}{\ln 2}; \quad \frac{dy}{dx} = \frac{1}{\ln 2} \cdot \frac{5}{5x} = \boxed{\frac{1}{x \ln 2}}$$

$$\text{(b)} \quad y = \frac{\ln(3x-1)}{\ln 10}; \quad \boxed{\frac{3}{(3x-1) \ln 10}}$$

$$\text{(c)} \quad y = \frac{\ln(2x^2-3)}{\ln b}; \quad \boxed{\frac{4x}{(2x^2-3) \ln b}}$$

$$\text{(d)} \quad y = \ln(x^4+3x^2); \quad \frac{4x^3+6x}{x^4+3x^2} = \boxed{\frac{2(2x^2+3)}{x(x^2+3)}}$$

$$\text{(e)} \quad y = \ln \sin 3x; \quad \frac{3 \cos 3x}{\sin 3x} = \boxed{3 \cot 3x}$$

$$\text{(f)} \quad y = \ln \sqrt{1+x^2} = \frac{1}{2} \ln(1+x^2);$$

$$\frac{1}{2} \cdot \frac{2x}{1+x^2} = \boxed{\frac{x}{1+x^2}}$$

Section 4 — Derivatives of exponential functions (Questions 10–12)

$$10 \text{ (a)} \quad \boxed{-3e^{5-3x}}$$

$$\text{(b)} \quad \boxed{(4x-1)e^{2x^2-x}}$$

$$\text{(c)} \quad \boxed{8e^{1+2x} - 6x}$$

$$\text{(d)} \quad 5e^{-2x}; \quad -10e^{-2x} = \boxed{-\frac{10}{e^{2x}}}$$

$$\text{(e)} \quad e^{3x} - e^{-3x}; \quad 3e^{3x} + 3e^{-3x} = \boxed{3e^{3x} + \frac{3}{e^{3x}}}$$

$$\text{(f)} \quad \frac{1}{3} \left(\frac{3}{2} \right) e^{\frac{3}{2}x} + 4 \left(-\frac{1}{2} \right) e^{-\frac{1}{2}x} = \boxed{\frac{1}{2} e^{\frac{3}{2}x} - 2e^{-\frac{1}{2}x}}$$

$$\text{(g)} \quad \boxed{2(x+1)e^{(x+1)^2}}$$

$$\text{(h)} \quad e^{-x^2}; \quad -2xe^{-x^2} = \boxed{-\frac{2x}{e^{x^2}}}$$

11 (a) Product rule: $e^x \sqrt{x} + e^x \left(\frac{1}{2\sqrt{x}} \right) = \frac{e^x(2x+1)}{2\sqrt{x}}$

(b) $-\frac{4}{x} = -4x^{-1}$; $4x^{-2}e^{-\frac{4}{x}} = \frac{4e^{-\frac{4}{x}}}{x^2}$

(c) $y = e^{-\frac{1}{3}x}$; $-\frac{1}{3}e^{-\frac{1}{3}x} = -\frac{1}{3\sqrt[3]{e^x}}$

(d) $y = e^{2x} + 2 + e^{-2x}$; $2e^{2x} - 2e^{-2x}$

(e) $y = e^{2x} + 3e^{-x}$; $2e^{2x} - 3e^{-x}$

(f) $y = e^{-x} + 2e^{-2x}$; $-e^{-x} - 4e^{-2x}$

12 (a) $4(2 - e^{-x})^3(e^{-x}) = 4e^{-x}(2 - e^{-x})^3$

(b) $-e^{1-x} \cos 3x + e^{1-x}(-3 \sin 3x)$
 $= -e^{1-x}(\cos 3x + 3 \sin 3x)$

(c) $\frac{1}{2}(1 + e^{2x})^{-\frac{1}{2}}(2e^{2x}) = \frac{e^{2x}}{\sqrt{1 + e^{2x}}}$

(d) $-3 \sin 3x e^{2+\cos 3x}$

(e) $\frac{\sin x(2e^{2x}) - e^{2x} \cos x}{\sin^2 x} = \frac{e^{2x}(2 \sin x - \cos x)}{\sin^2 x}$

Section 5 — Simple applications (Questions 13–15)

13 $\frac{dy}{dx} = 2pe^{2x} - qe^{-x}$, $\frac{d^2y}{dx^2} = 4pe^{2x} + qe^{-x}$
 $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 4pe^{2x} + qe^{-x} - 2pe^{2x} + qe^{-x}$
 $= 2pe^{2x} + 2qe^{-x}$

Comparing with $6e^{2x} + 4e^{-x}$: $2p = 6$, $2q = 4$

$p = 3$, $q = 2$

14(i) $\frac{dy}{dx} = 2e^{-3x} + 2x(-3)e^{-3x} = 2e^{-3x}(1 - 3x)$

14(ii)

$\frac{d^2y}{dx^2} = -6e^{-3x}(1 - 3x) + 2e^{-3x}(-3) = e^{-3x}(18x - 12)$

$\frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + 6y = e^{-3x}[(18x - 12) + (10 - 30x) + 12x]$
 $= -2e^{-3x}$

$q = e^{3x}(-2e^{-3x}) = -2$

15

$\frac{d}{dx} \left[5 \sin^2 \left(x + \frac{\pi}{4} \right) \right] = 5(2) \sin \left(x + \frac{\pi}{4} \right) \cos \left(x + \frac{\pi}{4} \right)$

$= 5 \sin \left(2x + \frac{\pi}{2} \right)$ (double angle formula)

$\sin 2A = 2 \sin A \cos A$

$= 5 \cos 2x$, since $\sin \left(2x + \frac{\pi}{2} \right) = \cos 2x$. $k = 5$

Section 6 — Tangents and normals (Questions 16–18)

16(a) $\frac{dy}{dx} = \cos x + \frac{1}{x}$

16(b) At $x = \frac{\pi}{2}$: $\frac{dy}{dx} = \cos \frac{\pi}{2} + \frac{2}{\pi} = \frac{2}{\pi}$,

$y = \sin \frac{\pi}{2} + \ln \frac{\pi}{2} = 1 + \ln \frac{\pi}{2}$

$y = \frac{2}{\pi}x + c$: $1 + \ln \frac{\pi}{2} = 1 + c$, $c = \ln \frac{\pi}{2}$

$y = \frac{2}{\pi}x + \ln \frac{\pi}{2}$

17(a) $e^{2x} = 4$; $2x = \ln 4$; $x = \frac{1}{2} \ln 4 = \ln 2$ (shown)

17(b) $\frac{dy}{dx} = 2e^{2x} = 2(4) = 8$

$y = 8x + c$: $4 = 8 \ln 2 + c$, $c = 4 - 8 \ln 2$

$y = 8x + 4 - 8 \ln 2$

17(c) Gradient of normal $= -\frac{1}{8}$

$4 = -\frac{1}{8} \ln 2 + c$, $c = 4 + \frac{1}{8} \ln 2$

$y = -\frac{1}{8}x + 4 + \frac{1}{8} \ln 2$

17(d) A (tangent, $x = 0$): $y = 4 - 8 \ln 2$

B (normal, $x = 0$): $y = 4 + \frac{1}{8} \ln 2$

$AB = \left(4 + \frac{1}{8} \ln 2 \right) - (4 - 8 \ln 2) = \frac{65}{8} \ln 2$

Distance of P from the y-axis $= \ln 2$

Area $= \frac{1}{2} \cdot \frac{65}{8} \ln 2 \cdot \ln 2 = \frac{65}{16} (\ln 2)^2$ units²

18 $\frac{d}{dx} \left(\frac{x}{x+2} \right) = \frac{(x+2) - x}{(x+2)^2} = \frac{2}{(x+2)^2}$

$\frac{dy}{dx} = \frac{2}{(x+2)^2} \div \frac{x}{x+2} = \frac{2}{x(x+2)}$

$2y + 8x = 5$: $y = -4x + \frac{5}{2}$, gradient -4 .

Gradient of normal $= -4$, so gradient of tangent $= \frac{1}{4}$.

$\frac{2}{x(x+2)} = \frac{1}{4}$; $x^2 + 2x - 8 = 0$; $(x+4)(x-2) = 0$

$x = 2$: $y = \ln \frac{2}{4} = -\ln 2$; $x = -4$: $y = \ln \frac{-4}{-2} = \ln 2$

$(2, -\ln 2)$ or $(-4, \ln 2)$

Section 7 — Rates of change (Questions 19–21)

19(a) $y = \frac{1}{3} \ln(3x^2 + 1)$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{6x}{3x^2 + 1} = \frac{2x}{3x^2 + 1}; \quad \boxed{k = 2}$$

19(b) $\frac{dy}{dt} = -3$. At $x = -1$: $\frac{dy}{dx} = \frac{-2}{4} = -\frac{1}{2}$

$$\frac{dx}{dt} = \frac{dy}{dt} \times \frac{dx}{dy} = -3 \times (-2) = \boxed{6 \text{ units/s}}$$

20 $\frac{dy}{dt} = 2$

$$\frac{dy}{dx} = \tan 2x + x(2 \sec^2 2x) = \tan 2x + 2x \sec^2 2x$$

At $x = \frac{\pi}{8}$: $\tan \frac{\pi}{4} = 1$, $\sec^2 \frac{\pi}{4} = 2$

$$\frac{dy}{dx} = 1 + 2 \left(\frac{\pi}{8} \right) (2) = 1 + \frac{\pi}{2} = \frac{2 + \pi}{2}$$

$$\frac{dx}{dt} = 2 \times \frac{2}{2 + \pi} = \boxed{\frac{4}{2 + \pi} \text{ units/s}}$$

21(a) $\tan \theta = \frac{x}{50}$; $\boxed{x = 50 \tan \theta}$

21(b) $\frac{dx}{d\theta} = 50 \sec^2 \theta$; $\frac{d\theta}{dt} = 0.1$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = 50 \sec^2 \theta \times 0.1 = 5 \sec^2 \theta$$

At $\theta = \frac{\pi}{3}$: $\sec^2 \frac{\pi}{3} = 4$; $\frac{dx}{dt} = \boxed{20 \text{ m/s}}$

Section 8 — Increasing and decreasing functions (Questions 22–24)

22

$$\begin{aligned} \frac{dy}{dx} &= 2e^{-x^2} + (2x - 1)(-2x)e^{-x^2} = 2e^{-x^2} (1 + x - 2x^2) \\ &= -2e^{-x^2} (2x + 1)(x - 1) \end{aligned}$$

Increasing: $\frac{dy}{dx} > 0$. Since $e^{-x^2} > 0$:

$$(2x + 1)(x - 1) < 0$$



$$\boxed{-\frac{1}{2} < x < 1}$$

23 $y = \frac{1}{2} \ln(2x - 1) - \frac{1}{2} \ln(2x + 1)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x - 1} - \frac{1}{2x + 1} = \frac{(2x + 1) - (2x - 1)}{(2x - 1)(2x + 1)} \\ &= \frac{2}{4x^2 - 1} \end{aligned}$$

For $x > \frac{1}{2}$, $4x^2 - 1 > 0$, so $\frac{dy}{dx} > 0$.

y is an increasing function. (proved)

24(a) $\frac{dy}{dx} = \frac{-2}{5 - 2x}$. The curve exists only where

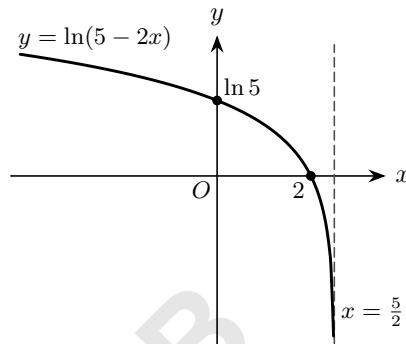
$$5 - 2x > 0, \text{ so } \frac{2}{5 - 2x} > 0 \text{ and } \frac{dy}{dx} < 0.$$

y is a decreasing function. (shown)

24(b) At $x = 0$: $y = \ln 5$, $\frac{dy}{dx} = -\frac{2}{5}$; gradient of normal $= \frac{5}{2}$

$$\boxed{y = \frac{5}{2}x + \ln 5}$$

24(c) Vertical asymptote $x = \frac{5}{2}$; $y = 0$ when $5 - 2x = 1$, i.e. $x = 2$; y -intercept $\ln 5$.



Section 9 — Stationary points, maxima and minima (Questions 25–27)

25 $\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x) = 0$

Since $e^{-x} \neq 0$: $x = 0$ or $x = 2$; $y = 0$ or $y = 4e^{-2}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) \\ &= e^{-x}(x^2 - 4x + 2) \end{aligned}$$

At $x = 0$: $2 > 0$, minimum. At $x = 2$: $-2e^{-2} < 0$, maximum.

$$\boxed{(0, 0)} \text{ minimum; } \boxed{\left(2, \frac{4}{e^2}\right)} \text{ maximum}$$

26 $\frac{dy}{dx} = 2 \ln x + 2x \left(\frac{1}{x} \right) - 5 = 2 \ln x - 3 = 0$

$$\ln x = \frac{3}{2}; \quad x = e^{\frac{3}{2}}; \quad y = 2e^{\frac{3}{2}} \left(\frac{3}{2} \right) - 5e^{\frac{3}{2}} = -2e^{\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x} > 0 \text{ for } x > 0$$

$$\boxed{\left(e^{\frac{3}{2}}, -2e^{\frac{3}{2}}\right) \approx (4.48, -8.96)}, \text{ a } \boxed{\text{minimum point}}$$

$$\begin{aligned}
\mathbf{27(a)} \quad & \text{Height} = 10 \sin \theta; \text{ top width} \\
& = 10 + 2(10 \cos \theta) = 10 + 20 \cos \theta \\
A & = \frac{1}{2} [10 + (10 + 20 \cos \theta)] (10 \sin \theta) \\
& = (10 + 10 \cos \theta)(10 \sin \theta) = 100 \sin \theta (1 + \cos \theta) \\
& \text{(shown)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{27(b)} \quad & \frac{dA}{d\theta} = 100 [\cos \theta (1 + \cos \theta) + \sin \theta (-\sin \theta)] \\
& = 100 (\cos \theta + \cos^2 \theta - \sin^2 \theta) \\
& = 100 (2 \cos^2 \theta + \cos \theta - 1) = 0
\end{aligned}$$

$$(2 \cos \theta - 1)(\cos \theta + 1) = 0; \quad \cos \theta = \frac{1}{2} \quad (\cos \theta = -1 \text{ rej})$$

$$\theta = \frac{\pi}{3}$$

$$\begin{aligned}
\mathbf{27(c)} \quad & A = 100 \sin \frac{\pi}{3} \left(1 + \cos \frac{\pi}{3}\right) \\
& = 100 \left(\frac{\sqrt{3}}{2}\right) \left(\frac{3}{2}\right) = \boxed{75\sqrt{3}}
\end{aligned}$$

$$\frac{dA}{d\theta} = 100(\cos \theta + \cos 2\theta); \quad \frac{d^2A}{d\theta^2} = 100(-\sin \theta - 2 \sin 2\theta)$$

$$\text{At } \theta = \frac{\pi}{3}: \quad 100 \left(-\frac{\sqrt{3}}{2} - \sqrt{3}\right) = -150\sqrt{3} < 0;$$

maximum

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