

Differentiation: Maxima and Minima

Increasing and Decreasing Functions, Stationary Points, Maxima and Minima

Secondary 4 — G3 Additional Mathematics

Notes

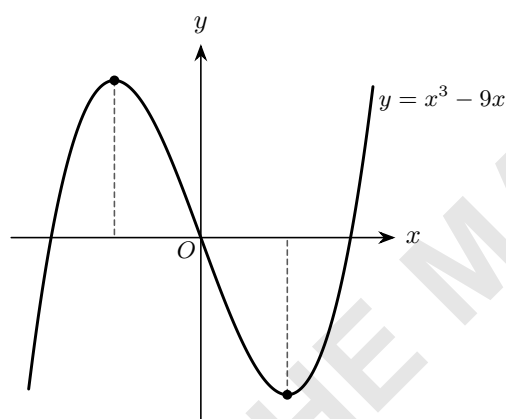
The second derivative

$\frac{d^2y}{dx^2}$ means $\frac{d}{dx} \left(\frac{dy}{dx} \right)$. To obtain $\frac{d^2y}{dx^2}$, we differentiate the expression of $\frac{dy}{dx}$ with respect to x again.

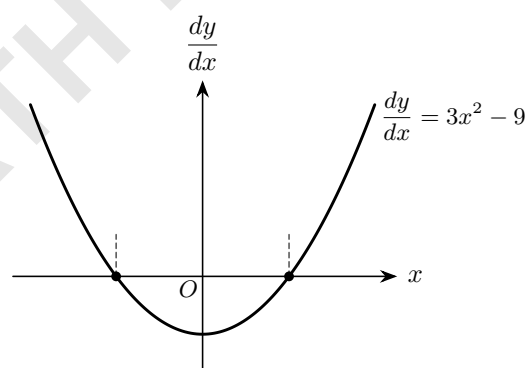
Note 1: $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx} \right)^2$

Note 2: $f'(x)$ denotes $\frac{dy}{dx}$ and $f''(x)$ denotes $\frac{d^2y}{dx^2}$.

Differentiating once gives the **gradient function** $\frac{dy}{dx}$. That function has a graph and a gradient of its own, and that gradient is $\frac{d^2y}{dx^2}$: the rate at which the gradient of the curve changes as x increases.



the curve: turning points
where the gradient is 0



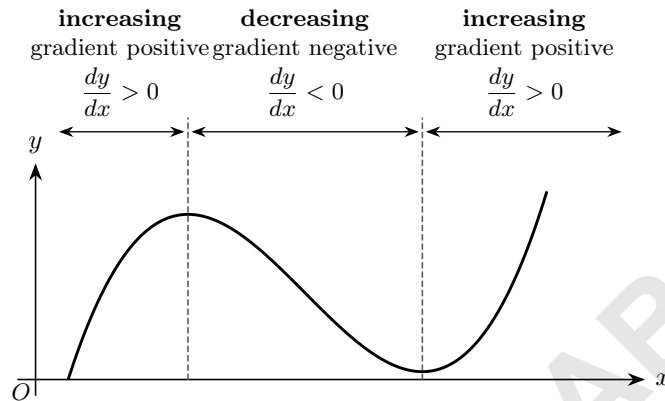
the gradient function; its own

gradient is $\frac{d^2y}{dx^2} = 6x$

Increasing and decreasing functions

If y is an **increasing function** (i.e. y increases as x increases), the gradient is positive. $\therefore \frac{dy}{dx} > 0$

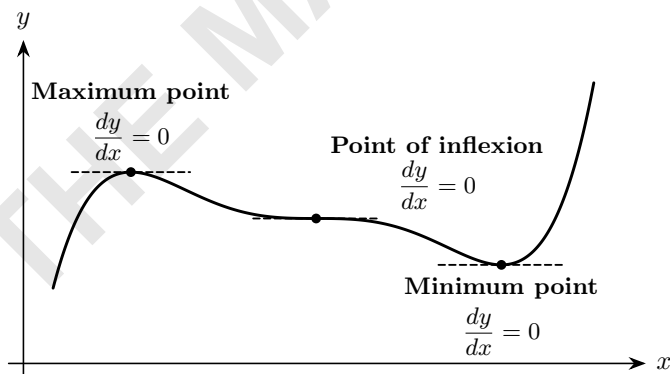
If y is a **decreasing function** (i.e. y decreases as x increases), the gradient is negative. $\therefore \frac{dy}{dx} < 0$



Stationary points

Stationary points are at $\frac{dy}{dx} = 0$.

Stationary points can be **maximum points**, **minimum points** or **points of inflexion**.

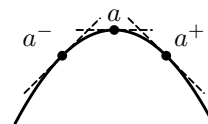


- 1) There are 3 types of stationary points: **maximum points**, **minimum points** and **points of inflexion**.
- 2) Stationary points have gradient = 0 (their tangents are horizontal lines).
- 3) $\frac{dy}{dx} = 0$ at all stationary points.
- 4) Maximum and minimum points are also called **turning points**.
- 5) There are 2 methods to determine the **nature** of a stationary point: (i) the first derivative test, (ii) the second derivative test.

To use the first derivative test, let the stationary point be at $x = a$. Here a^- means a value of x slightly less than a (immediately to the left), and a^+ a value slightly more than a (immediately to the right).

Case 1: Maximum point. If the gradient of the tangent to the immediate **left** of the stationary point is **positive** and the gradient of the tangent to the immediate **right** of the stationary point is **negative**, the stationary point is a **maximum point**.

Value of x	$x = a^-$	$x = a$	$x = a^+$
Sign of $\frac{dy}{dx}$	Positive (+)	0	Negative (-)
Slope	/	—	\



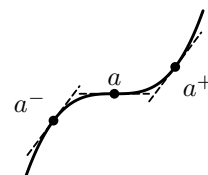
Case 2: Minimum point. If the gradient of the tangent to the immediate **left** of the stationary point is **negative** and the gradient of the tangent to the immediate **right** of the stationary point is **positive**, the stationary point is a **minimum point**.

Value of x	$x = a^-$	$x = a$	$x = a^+$
Sign of $\frac{dy}{dx}$	Negative (-)	0	Positive (+)
Slope	\	—	/

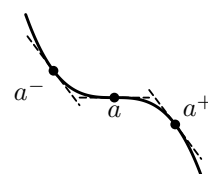


Case 3: Point of inflexion. If the gradient of the tangent to the immediate **left** of the stationary point has the **same sign** as the gradient of the tangent to the immediate **right**, the stationary point is a **point of inflexion**.

Value of x	$x = a^-$	$x = a$	$x = a^+$
Sign of $\frac{dy}{dx}$	Positive (+)	0	Positive (+)
Slope	/	—	/



Value of x	$x = a^-$	$x = a$	$x = a^+$
Sign of $\frac{dy}{dx}$	Negative (-)	0	Negative (-)
Slope	\	—	\



The second derivative test

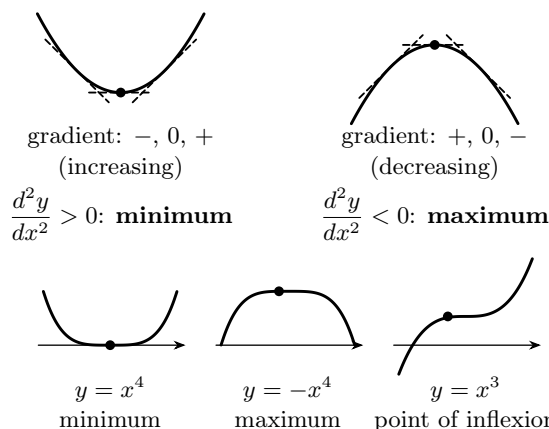
After finding the stationary points, we can also determine the nature of the stationary points using the second derivative test:

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, the point is a **minimum point**.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the point is a **maximum point**.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the nature **cannot be determined**, i.e. the test is **inconclusive**. Proceed to use the first derivative test to determine the nature.

Why the test works: $\frac{d^2y}{dx^2}$ is the rate of change of the gradient. At a minimum point the gradient goes from negative, through 0, to positive, so the gradient is increasing; at a maximum point it goes from positive, through 0, to negative, so it is decreasing.



each has $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at $x = 0$: the second derivative test cannot tell them apart

Maxima and minima in practical problems

Determining maxima and minima in practical situations is similar to finding maximum and minimum points on graphs.

Guidelines

- 1) Read the problem carefully.
- 2) Construct a mathematical equation to connect the variables in the problem (if the equation is not given in the question).
- 3) Maxima and minima are both stationary points. Regardless of which one we are finding, we first find all the stationary points by solving $f'(x) = 0$.
- 4) After finding the stationary points, determine or verify the nature of the stationary points using either the first or the second derivative test.

Mensuration formulae you must know (they are not given in tests or examinations)

Circle	area πr^2 , circumference $2\pi r$	Cylinder	$V = \pi r^2 h$, curved surface area $2\pi r h$
Triangle	area $\frac{1}{2}ab \sin C$	Cone	$V = \frac{1}{3}\pi r^2 h$, curved surface area $\pi r l$
Sphere	$V = \frac{4}{3}\pi r^3$, surface area $4\pi r^2$	Hemisphere	$V = \frac{2}{3}\pi r^3$, curved surface area $2\pi r^2$

Exercises

Section 1: The Second Derivative

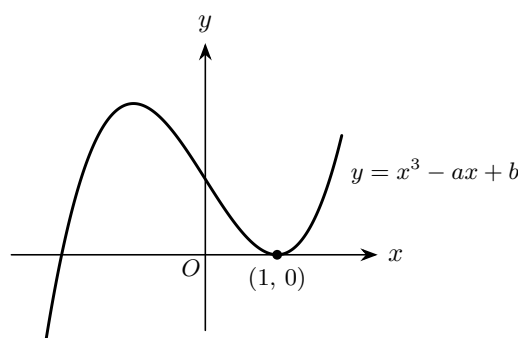
1. It is given that $y = x^2 - 2x + 5$. Solve the equation $\left(\frac{dy}{dx}\right)^2 + 4\left(\frac{d^2y}{dx^2}\right) = 4y + 2x$.
2. Given that $y = \frac{x}{2x-1}$, where $x \neq \frac{1}{2}$, show that $(2x-1)\left(\frac{d^2y}{dx^2}\right) + 4\frac{dy}{dx} = 0$.
3. For the curve $y = x^3 - 6x^2 + 5$, find the range of values of x for which $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are both negative.

Section 2: Increasing and Decreasing Functions

4. Find the range of values of x for which the function $y = 4x^3 - 6x^2 - 9x + 2$ is increasing.
5. A curve has the equation $y = f(x)$, where $f(x) = 2x^3 + 6x^2 + 9x - 4$.
 - (a) Express $f'(x)$ in the form $a(x+b)^2 + c$, where a , b and c are constants.
 - (b) Deduce that $f(x)$ is an increasing function.
 - (c) Find the set of values of x for which $f'(x) + 3x \leq f''(x)$.
6. The curve $y = f(x)$ is defined for $x > 3$ by $f(x) = \frac{x^2 + 7}{x - 3}$.
 - (a) Obtain an expression for $f'(x)$.
 - (b) Find the range of values of x for which f is an increasing function.
 - (c) Showing full working, determine whether the **gradient** of the curve is an increasing or a decreasing function.

Section 3: Stationary Points and the First Derivative Test

7. Find the coordinates of the stationary point on the curve $y = x^3 - 6x^2 + 12x + 5$ and use the first derivative test to determine its nature.
8. The diagram shows part of the curve $y = x^3 - ax + b$, where a and b are positive constants. The curve has a minimum point at $(1, 0)$. Find
 - (a) the values of a and b ,
 - (b) the coordinates of the maximum point of the curve.

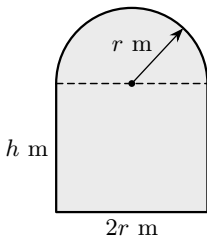
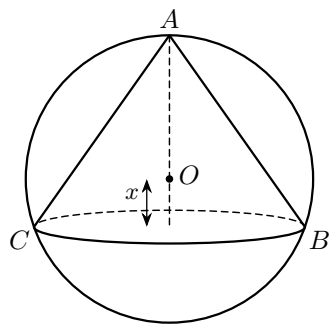


9. Find the stationary points of the curve $y = x^4 - 8x^2 + 3$ and determine the nature of each one.

Section 4: The Second Derivative Test

10. The equation of a curve is $y = 2\sqrt{x} + \frac{8}{x}$, for $x > 0$.
- Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
 - Show that the curve has a stationary point when $x = 4$.
 - Use the second derivative test to determine the nature of this stationary point.
11. The equation of a curve is $y = 3 - x - \frac{2x + 6}{x + 1}$.
- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
 - Find the x -coordinate(s) of the stationary point(s) of the curve.
 - Determine the nature of each stationary point.
12. Two positive variables x and y are connected by the equation $x^2y - 5x^2 - 4 = 0$.
- Find the least value of $(x + y)$.
 - Verify that this value of $(x + y)$ is a minimum.

Section 5: Applications of Maxima and Minima

13. A window frame is made from a metal strip 6 m long. The frame is in the shape of a rectangle of width $2r$ m and height h m, with a semicircle of radius r m on top. The strip runs along the two sides, the bottom and the curved top, but not along the join between the rectangle and the semicircle. The area of the window is A m².
- 
- Express h in terms of r .
 - Show that $A = 6r - \left(2 + \frac{\pi}{2}\right)r^2$.
 - Find the value of r for which A has a stationary value.
 - Find this value of A and determine whether it is a maximum or a minimum.
14. A lorry makes a journey of 300 km at a steady speed of v km/h. The running costs of the lorry, for fuel and the driver's wages, are $\$ \left(16 + \frac{v^2}{100}\right)$ per hour.
- Show that the total cost of the journey, $\$C$, is given by $C = \frac{4800}{v} + 3v$.
 - Find the speed at which the cost of the journey is least, and the least cost.
15. A right circular cone is inscribed in a sphere with centre O and radius 3 cm. The vertex A of the cone and the circumference of its base lie on the sphere, and O lies on the axis of the cone. The base of the cone is x cm from O , on the opposite side of O from A .
- 
- Show that the volume of the cone is $\frac{\pi}{3} (27 + 9x - 3x^2 - x^3)$ cm³.
 - Hence find the value of x for which the volume of the cone is a maximum, and find this maximum volume.

Solutions to Exercises

Section 1 — The second derivative (Questions 1–3)

1 $\frac{dy}{dx} = 2x - 2$, $\frac{d^2y}{dx^2} = 2$

$(2x - 2)^2 + 4(2) = 4(x^2 - 2x + 5) + 2x$

$4x^2 - 8x + 4 + 8 = 4x^2 - 8x + 20 + 2x$

$12 = 2x + 20$; $2x = -8$

$x = -4$

2 $\frac{dy}{dx} = \frac{1(2x - 1) - x(2)}{(2x - 1)^2} = \frac{-1}{(2x - 1)^2} = -(2x - 1)^{-2}$

$\frac{d^2y}{dx^2} = 2(2x - 1)^{-3}(2) = \frac{4}{(2x - 1)^3}$

$(2x - 1) \left(\frac{d^2y}{dx^2} \right) + 4 \frac{dy}{dx} = \frac{4}{(2x - 1)^2} - \frac{4}{(2x - 1)^2} = 0$

(shown)

3 $\frac{dy}{dx} = 3x^2 - 12x = 3x(x - 4)$, $\frac{d^2y}{dx^2} = 6x - 12$

$\frac{dy}{dx} < 0$: $3x(x - 4) < 0$



$0 < x < 4$

$\frac{d^2y}{dx^2} < 0$: $6x - 12 < 0$, $x < 2$

Both negative: $0 < x < 2$

Section 2 — Increasing and decreasing functions (Questions 4–6)

4 $\frac{dy}{dx} = 12x^2 - 12x - 9 > 0$

$3(2x + 1)(2x - 3) > 0$



$x < -\frac{1}{2}$ or $x > \frac{3}{2}$

5(a) $f'(x) = 6x^2 + 12x + 9 = 6(x^2 + 2x) + 9$

$= 6[(x + 1)^2 - 1] + 9 = 6(x + 1)^2 + 3$

5(b) Since $(x + 1)^2 \geq 0$, $6(x + 1)^2 + 3 \geq 3 > 0$

$f'(x) > 0$, so $f(x)$ is an increasing function. (shown)

5(c) $f''(x) = 12x + 12$

$6x^2 + 12x + 9 + 3x \leq 12x + 12$

$6x^2 + 3x - 3 \leq 0$; $3(2x - 1)(x + 1) \leq 0$



$-1 \leq x \leq \frac{1}{2}$

6(a) $f'(x) = \frac{2x(x - 3) - 1(x^2 + 7)}{(x - 3)^2} = \frac{x^2 - 6x - 7}{(x - 3)^2}$

6(b) $f'(x) > 0$. Since $(x - 3)^2 > 0$, $x^2 - 6x - 7 > 0$
 $(x - 7)(x + 1) > 0$; $x < -1$ or $x > 7$

Given $x > 3$: $x > 7$

6(c) Let $u = x^2 - 6x - 7$, $v = (x - 3)^2$: $\frac{du}{dx} = 2x - 6$,

$\frac{dv}{dx} = 2(x - 3)$.

$f''(x) = \frac{(x - 3)^2(2x - 6) - (x^2 - 6x - 7)(2)(x - 3)}{(x - 3)^4}$
 $= \frac{2[(x - 3)^2 - (x^2 - 6x - 7)]}{(x - 3)^3} = \frac{32}{(x - 3)^3}$

Given $x > 3$, $(x - 3)^3 > 0$, so $f''(x) > 0$.

The gradient is an increasing function.

Section 3 — First derivative test (Questions 7–9)

7 $\frac{dy}{dx} = 3x^2 - 12x + 12 = 3(x - 2)^2$

When $\frac{dy}{dx} = 0$: $x = 2$; $y = 8 - 24 + 24 + 5 = 13$

x	2^-	2	2^+
Sign of $\frac{dy}{dx}$	+	0	+
Slope	/	—	/

$(2, 13)$ is a point of inflexion.

8(a) $\frac{dy}{dx} = 3x^2 - a$

At $(1, 0)$, $\frac{dy}{dx} = 0$: $3 - a = 0$; $a = 3$

$(1, 0)$ on the curve: $0 = 1 - 3 + b$; $b = 2$

8(b) $\frac{dy}{dx} = 3x^2 - 3 = 3(x - 1)(x + 1) = 0$: the other stationary point is at $x = -1$.

$y = (-1)^3 - 3(-1) + 2 = 4$

x	-1^-	-1	-1^+
Sign of $\frac{dy}{dx}$	+	0	—
Slope	/	—	\

Maximum point $(-1, 4)$

9 $\frac{dy}{dx} = 4x^3 - 16x = 4x(x - 2)(x + 2)$

When $\frac{dy}{dx} = 0$: $x = 0$, $x = 2$ or $x = -2$

When $x = 0$, $y = 3$; when $x = \pm 2$,
 $y = 16 - 32 + 3 = -13$

x	-2^-	-2	-2^+
Sign of $\frac{dy}{dx}$	—	0	+
Slope	\	—	/

x	0^-	0	0^+
Sign of $\frac{dy}{dx}$	$+$	0	$-$
Slope	$\nearrow$	$-$	$\searrow$

x	2^-	2	2^+
Sign of $\frac{dy}{dx}$	$-$	0	$+$
Slope	$\searrow$	$-$	$\nearrow$

$(-2, -13)$ and $(2, -13)$ minimum points; $(0, 3)$ maximum point

Section 4 — Second derivative test (Questions 10–12)

10(i) $y = 2x^{\frac{1}{2}} + 8x^{-1}$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 8x^{-2} = \frac{1}{\sqrt{x}} - \frac{8}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} + 16x^{-3}$$

10(ii) When $x = 4$: $\frac{dy}{dx} = \frac{1}{2} - \frac{8}{16} = 0$

So the curve has a stationary point when $x = 4$. (shown)

10(iii) When $x = 4$: $\frac{d^2y}{dx^2} = -\frac{1}{2} \left(\frac{1}{8} \right) + \frac{16}{64} = \frac{3}{16} > 0$

The stationary point $(4, 6)$ is a minimum point.

11(i)

$$\frac{dy}{dx} = -1 - \frac{2(x+1) - 1(2x+6)}{(x+1)^2} = -1 + \frac{4}{(x+1)^2}$$

$$\frac{d^2y}{dx^2} = 4(-2)(x+1)^{-3} = -\frac{8}{(x+1)^3}$$

11(ii) $\frac{4}{(x+1)^2} = 1$; $(x+1)^2 = 4$; $x+1 = \pm 2$

$x = 1$ or $x = -3$

11(iii) At $x = 1$: $\frac{d^2y}{dx^2} = -\frac{8}{8} = -1 < 0$:

maximum point

At $x = -3$: $\frac{d^2y}{dx^2} = -\frac{8}{-8} = 1 > 0$: minimum point

12(i) $x^2y = 5x^2 + 4$; $y = 5 + \frac{4}{x^2}$

Let $z = x + y = x + 5 + 4x^{-2}$.

$$\frac{dz}{dx} = 1 - 8x^{-3}$$

When $\frac{dz}{dx} = 0$: $x^3 = 8$; $x = 2$

$y = 5 + \frac{4}{4} = 6$; $z = 2 + 6$

Least value of $x + y = 8$

12(ii) $\frac{d^2z}{dx^2} = 24x^{-4}$; at $x = 2$: $\frac{24}{16} = \frac{3}{2} > 0$

So 8 is indeed a minimum. (shown)

Section 5 — Applications (Questions 13–15)

13(a) Strip: two sides h , the bottom $2r$ and the arc πr :

$$2h + 2r + \pi r = 6; \quad h = \frac{6 - 2r - \pi r}{2}$$

13(b) $A = \text{rectangle} + \text{semicircle} = 2rh + \frac{1}{2}\pi r^2$
 $= r(6 - 2r - \pi r) + \frac{1}{2}\pi r^2$
 $= 6r - 2r^2 - \frac{1}{2}\pi r^2 = 6r - \left(2 + \frac{\pi}{2}\right)r^2$ (shown)

13(c) $\frac{dA}{dr} = 6 - 2\left(2 + \frac{\pi}{2}\right)r = 6 - (4 + \pi)r = 0$

$$r = \frac{6}{4 + \pi} = 0.840 \quad (3 \text{ s.f.})$$

13(d) $A = 6\left(\frac{6}{4 + \pi}\right) - \frac{4 + \pi}{2}\left(\frac{6}{4 + \pi}\right)^2 = \frac{18}{4 + \pi} =$

$$2.52 \text{ m}^2$$

$$\frac{d^2A}{dr^2} = -(4 + \pi) = -7.14 < 0$$
: a maximum.

14(i) Time taken $= \frac{300}{v}$ hours.

$$C = \frac{300}{v} \left(16 + \frac{v^2}{100}\right) = \frac{4800}{v} + 3v \quad (\text{shown})$$

14(ii) $\frac{dC}{dv} = -\frac{4800}{v^2} + 3 = 0$; $v^2 = 1600$
 $v = 40$ (–40 N.A.)

$$\frac{d^2C}{dv^2} = \frac{9600}{v^3} > 0$$
: minimum.

$$C = 120 + 120 = 240$$

Least cost \$240 at 40 km/h

15(a) Let M be the centre of the base and r cm its radius. $OB = 3$ (radius of the sphere), $OM = x$.

$$r^2 = OB^2 - OM^2 = 9 - x^2$$
; height

$$AM = AO + OM = 3 + x$$

$$V = \frac{1}{3}\pi(9 - x^2)(3 + x) = \frac{\pi}{3}(27 + 9x - 3x^2 - x^3)$$

(shown)

15(b) $\frac{dV}{dx} = \frac{\pi}{3}(9 - 6x - 3x^2) = -\pi(x^2 + 2x - 3) = 0$

$$(x+3)(x-1) = 0$$
; $x = 1$ or $x = -3$ (N.A.)

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-6 - 6x) = -4\pi < 0$$
 at $x = 1$: maximum.

$$x = 1$$
; $V = \frac{\pi}{3}(27 + 9 - 3 - 1) = \frac{32\pi}{3} \text{ cm}^3$

$$(33.5 \text{ cm}^3)$$