

Differentiation: Rates of Change

Rates of Change, Connected Rates of Change and their Applications

Secondary 4 — G3 Additional Mathematics

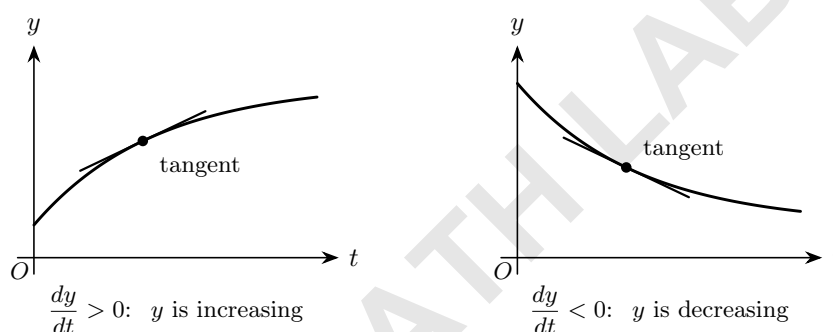
Notes

Rate of change

$\frac{dy}{dx}$ means the rate of change of y with respect to x .

$\frac{dy}{dt}$ means the rate of change of y (with respect to time t).

When a quantity y changes with time, $\frac{dy}{dt}$ at an instant is the **gradient of the tangent** to the graph of y against t at that instant. Its sign tells the direction of the change: a positive rate means y is increasing, and a negative rate means y is decreasing.



Connected rates of change

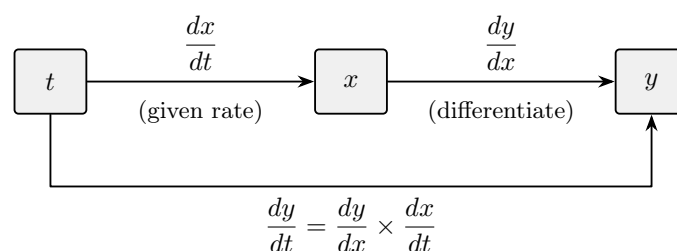
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

in which $\frac{dy}{dt}$ means the rate of change of y (with respect to time),

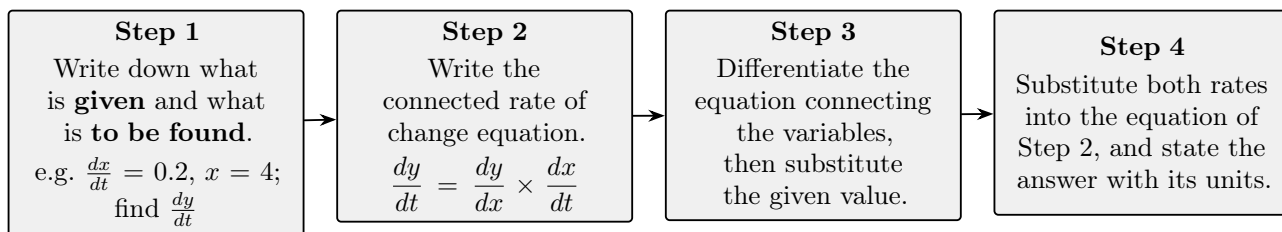
$\frac{dx}{dt}$ means the rate of change of x (with respect to time)

and $\frac{dy}{dx}$ means the rate of change of y (with respect to x).

The variable x changes with time, and y changes with x . The rate at which x changes is usually given; the rate at which y changes with x comes from differentiating the equation that connects them. Multiplying the two links gives the rate at which y changes with time.



Method. Every connected-rates question is worked in the same four steps.



In a practical problem the variables are lengths, areas and volumes. Read off the values given in the problem, write down what is to be found, and use the mensuration formula that connects the quantities (assumed knowledge).

Formulae you will need

Circle	$A = \pi r^2$, circumference $2\pi r$	Cylinder	$V = \pi r^2 h$, total surface area $2\pi r^2 + 2\pi r h$
Sector	arc $s = r\theta$, $A = \frac{1}{2}r^2\theta$	Cone	$V = \frac{1}{3}\pi r^2 h$, curved surface area $\pi r l$
Sphere	$V = \frac{4}{3}\pi r^3$, surface area $4\pi r^2$	Cube	$V = x^3$, total surface area $6x^2$
Triangle	$A = \frac{1}{2}ab \sin C$	Cuboid	$V = lbh$

Signs of rates. A quantity that is increasing (filling, growing, expanding) has a **positive** rate of change. A quantity that is decreasing (leaking, melting, cooling) has a **negative** rate of change: an area decreasing at $64 \text{ cm}^2/\text{s}$ means $\frac{dA}{dt} = -64$. When a question asks for a *rate of decrease*, state it as a positive number: $\frac{dr}{dt} = -\frac{1}{\pi}$ means the radius is decreasing at $\frac{1}{\pi} \text{ cm/s}$.

Two variables in the formula. The volume of a cone, $V = \frac{1}{3}\pi r^2 h$, contains two variables, so it cannot be differentiated with respect to h as it stands. Use **similar triangles** to write r in terms of h first; then V is a function of h alone.

Exercises

Section 1: Rates of Change (Basics)

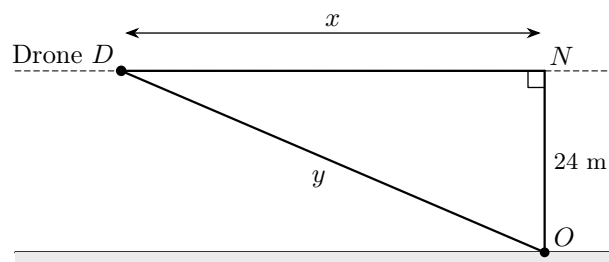
1. A valve at the bottom of a container is opened. The volume of oil, $V \text{ cm}^3$, left in the container t seconds later is $V = 2(20 - t)^3$ for $0 \leq t \leq 20$. Find the rate at which oil is draining out of the container when $t = 5$.
2. A cup of hot tea is left to cool in a room. Its temperature, $\theta^\circ\text{C}$, t minutes later is given by $\theta = 20 + \frac{60}{t+2}$.
 - (a) Find the temperature of the tea when it is first left to cool.
 - (b) Find the rate at which the tea is cooling when $t = 3$.
3. A triangle has base $2x \text{ cm}$ and height $(x - 1) \text{ cm}$. The value of x at time t seconds is given by $x = 2 + 4t$ ($t \geq 0$).
 - (a) Show that the area, $A \text{ cm}^2$, of the triangle in terms of t is $A = 16t^2 + 12t + 2$.
 - (b) Hence, find the rate of change of the area at the instant when $t = 3$.

Section 2: Connected Rates of Change

4. The variables x and y are connected by $y = 2x^3 + x^2$. Given that y is increasing at 6 units per second, find the rate of change of x at the instant when $x = 1$.
5. The variables x and y are connected by $\frac{3}{y} = \frac{1}{x} + 2$. Given that x is increasing at 0.6 units per second, find the rate of change of y at the instant when $x = 1$.
6. The variables x and y are connected by $y = \sqrt{x^2 + 9}$, and both vary with time. Find the value of x at the instant when the rate of change of y is 0.8 times the rate of change of x .

Section 3: Applications of Connected Rates of Change

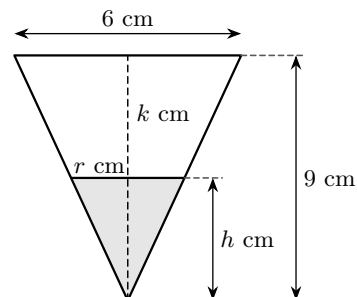
7. A spherical snowball melts at a constant rate, keeping its spherical shape. Its volume is $288\pi \text{ cm}^3$ at the start and $204\pi \text{ cm}^3$ after 3 minutes. Find the rate at which its radius is decreasing at the time $t = 9$ minutes.
8. In the diagram, a drone D flies horizontally at a constant height of 24 m and at a speed of 5 m/s towards the point N , which is directly above an observer O standing on level ground. Given that $DN = x \text{ m}$ and $DO = y \text{ m}$,
 - (a) express y in terms of x ,
 - (b) find the rate of change of the distance between the drone and the observer when $x = 32$.



9. Under a heating process, the length of each edge of a metal cube increases at a constant rate. The initial length of each edge is 3 cm, and it increases at 0.5 cm/s.
- Find the rate of increase of the volume of the cube after 6 seconds.
 - Hence, find the corresponding rate of increase of the total surface area of the cube.

Section 4: Applications Using Similar Triangles

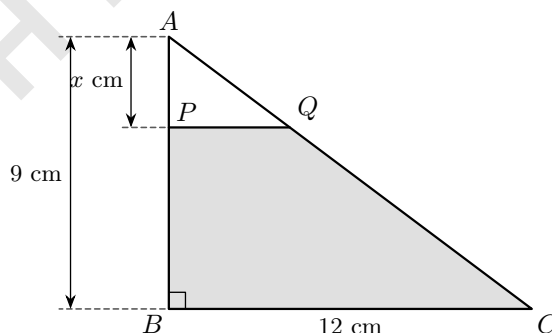
10. A cocktail glass is in the shape of an inverted right circular cone. Its circular rim has a diameter of 6 cm and its depth is 9 cm. Juice is poured into the empty glass at a constant rate of 3 cm^3 per second. After t seconds, the volume of juice in the glass is $V \text{ cm}^3$, the surface of the juice has radius $r \text{ cm}$ and the depth of the juice is $h \text{ cm}$. As the juice rises, the distance, $k \text{ cm}$, between the surface of the juice and the rim of the glass decreases.



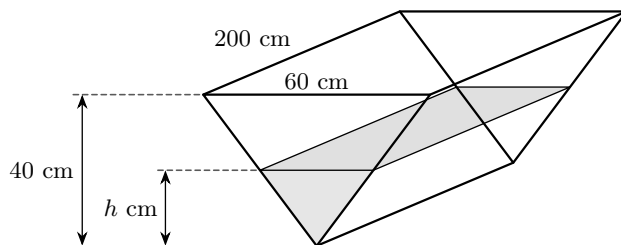
- Show that $V = \frac{\pi h^3}{27}$.
 - Find the rate of change of h when $h = 3$.
 - Hence, find the rate of decrease of k at this instant.
11. The diagram shows a triangle ABC , right-angled at B , with $AB = 9 \text{ cm}$ and $BC = 12 \text{ cm}$. A line PQ , with P on AB and Q on AC , moves away from A at a steady rate of 0.6 cm/s, keeping parallel to BC .

When $AP = x \text{ cm}$,

- show that $PQ = \frac{4x}{3} \text{ cm}$,
- find the area, $A \text{ cm}^2$, of the shaded region $PBCQ$ in terms of x ,
- find the rate of change of the shaded area at the instant when $AP = 6 \text{ cm}$.



12. The diagram shows a water trough 200 cm long. Its cross-section is an isosceles triangle with the vertex at the bottom; the width across the top is 60 cm and the depth is 40 cm. Water flows into the trough at a constant rate of $3000 \text{ cm}^3/\text{s}$. The depth of the water is $h \text{ cm}$. Find the rate at which the depth of the water is increasing at the instant when the width of the water surface is 30 cm.



Solutions to Exercises

Section 1 — Rates of change (Questions 1–3)

1 $V = 2(20 - t)^3$

$$\frac{dV}{dt} = 2(3)(20 - t)^2(-1) = -6(20 - t)^2$$

Sub $t = 5$: $\frac{dV}{dt} = -6(15)^2 = -1350$

Oil drains out at $\boxed{1350 \text{ cm}^3/\text{s}}$.

2(a) When first left, $t = 0$: $\theta = 20 + \frac{60}{0 + 2} = 50$

The temperature is $\boxed{50^\circ\text{C}}$.

2(b) $\theta = 20 + 60(t + 2)^{-1}$

$$\frac{d\theta}{dt} = -60(t + 2)^{-2}$$

Sub $t = 3$: $\frac{d\theta}{dt} = -60(5)^{-2} = -2.4$

The tea is cooling at $\boxed{2.4^\circ\text{C per minute}}$.

3(a) $A = \frac{1}{2}(2x)(x - 1) = x(x - 1)$. Sub $x = 2 + 4t$:

$$A = (2 + 4t)(1 + 4t) = 2 + 8t + 4t + 16t^2$$

$$A = 16t^2 + 12t + 2 \quad (\text{shown})$$

3(b) $\frac{dA}{dt} = 32t + 12$

Sub $t = 3$: $\frac{dA}{dt} = 96 + 12 = \boxed{108 \text{ cm}^2/\text{s}}$

Section 2 — Connected rates of change (Questions 4–6)

4 Given: $\frac{dy}{dt} = 6$, $x = 1$. Find $\frac{dx}{dt}$.

$$\frac{dy}{dx} = 6x^2 + 2x; \text{ sub } x = 1: \frac{dy}{dx} = 8$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}; \quad 6 = 8 \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = \boxed{\frac{3}{4} \text{ units/s}}$$

5 $\frac{3}{y} = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$, so $y = \frac{3x}{1 + 2x}$.

$u = 3x$	$v = 1 + 2x$
$\frac{du}{dx} = 3$	$\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{(1 + 2x)(3) - 3x(2)}{(1 + 2x)^2} = \frac{3}{(1 + 2x)^2}$$

Sub $x = 1$: $\frac{dy}{dx} = \frac{3}{9} = \frac{1}{3}$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{1}{3} \times 0.6 = \boxed{0.2 \text{ units/s}}$$

6 Given $\frac{dy}{dt} = 0.8 \times \frac{dx}{dt}$. By comparison with

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}, \quad \frac{dy}{dx} = 0.8.$$

$$y = (x^2 + 9)^{0.5}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 9)^{-0.5}(2x) = \frac{x}{\sqrt{x^2 + 9}}$$

$$\frac{x}{\sqrt{x^2 + 9}} = 0.8. \text{ The left side must be positive, so } x > 0.$$

$$x^2 = 0.64(x^2 + 9); \quad 0.36x^2 = 5.76; \quad x^2 = 16$$

$$x = 4 \quad (x = -4 \text{ gives } \frac{dy}{dx} = -0.8, \text{ rejected}). \quad \boxed{x = 4}$$

Section 3 — Applications (Questions 7–9)

7 The volume falls by $288\pi - 204\pi = 84\pi \text{ cm}^3$ in 3 minutes at a constant rate:

$$\frac{dV}{dt} = -28\pi \text{ cm}^3/\text{min}$$

At $t = 9$: $V = 288\pi - 9(28\pi) = 36\pi$

$$\frac{4}{3}\pi r^3 = 36\pi; \quad r^3 = 27; \quad r = 3$$

$$\frac{dV}{dr} = 4\pi r^2 = 36\pi$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}; \quad -28\pi = 36\pi \times \frac{dr}{dt}; \quad \frac{dr}{dt} = -\frac{7}{9}$$

The radius is decreasing at $\boxed{\frac{7}{9} \text{ cm/min}}$.

8(a) Triangle DNO is right-angled at N :

$$y^2 = x^2 + 24^2; \quad \boxed{y = \sqrt{x^2 + 576}}$$

8(b) The drone flies towards N , so x is decreasing:

$$\frac{dx}{dt} = -5.$$

$$y = (x^2 + 576)^{0.5}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 576)^{-0.5}(2x) = \frac{x}{\sqrt{x^2 + 576}}$$

Sub $x = 32$: $\sqrt{1024 + 576} = \sqrt{1600} = 40$, so

$$\frac{dy}{dx} = \frac{32}{40} = 0.8$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 0.8 \times (-5) = \boxed{-4 \text{ m/s}}$$

The distance between the drone and the observer is decreasing at 4 m/s.

9(a) Let the edge be $x \text{ cm}$: $x = 3 + 0.5t$, $\frac{dx}{dt} = 0.5$.

After 6 s: $x = 3 + 0.5(6) = 6$

$$V = x^3; \quad \frac{dV}{dx} = 3x^2 = 108$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 108 \times 0.5 = \boxed{54 \text{ cm}^3/\text{s}}$$

9(b) At the same instant, $x = 6$.

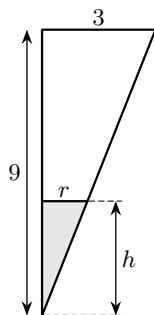
$$S = 6x^2; \quad \frac{dS}{dx} = 12x = 72$$

$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt} = 72 \times 0.5 = \boxed{36 \text{ cm}^2/\text{s}}$$

Section 4 — Similar triangles (Questions 10–12)

10(a) The rim has radius 3 and the depth is 9.

Using similar triangles: $\frac{r}{h} = \frac{3}{9}$; $r = \frac{h}{3}$



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27} \quad (\text{shown})$$

10(b) Given: $\frac{dV}{dt} = 3$, $h = 3$. Find $\frac{dh}{dt}$.

$$\frac{dV}{dh} = \frac{3\pi h^2}{27} = \frac{\pi h^2}{9}$$

$$\text{Sub } h = 3: \quad \frac{dV}{dh} = \pi$$

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 3 \div \pi = \frac{3}{\pi}$$

$$\frac{dh}{dt} = \boxed{0.955 \text{ cm/s}} \quad (3 \text{ s.f.})$$

10(c) $k = 9 - h$, so $\frac{dk}{dh} = -1$.

$$\frac{dk}{dt} = \frac{dk}{dh} \times \frac{dh}{dt} = -\frac{3}{\pi}$$

The rate of decrease of k is $\boxed{0.955 \text{ cm/s}} \quad (3 \text{ s.f.})$.

11(a) Triangles APQ and ABC are similar:

$$\frac{PQ}{BC} = \frac{AP}{AB}; \quad \frac{PQ}{12} = \frac{x}{9}; \quad PQ = \frac{12x}{9} = \frac{4x}{3} \quad (\text{shown})$$

11(b) $A = \text{area of } ABC - \text{area of } APQ$

$$A = \frac{1}{2}(12)(9) - \frac{1}{2}\left(\frac{4x}{3}\right)(x) = \boxed{54 - \frac{2}{3}x^2}$$

11(c) PQ moves away from A , so $\frac{dx}{dt} = 0.6$. $AP = 6$:

$x = 6$.

$$\frac{dA}{dx} = -\frac{4x}{3}; \quad \text{sub } x = 6: \quad \frac{dA}{dx} = -8$$

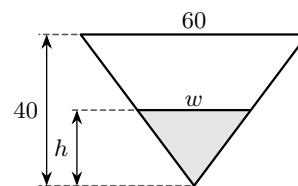
$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = -8 \times 0.6 = \boxed{-4.8 \text{ cm}^2/\text{s}}$$

The shaded area is decreasing at $4.8 \text{ cm}^2/\text{s}$.

12 Let the water surface have width w cm. Given

$$\frac{dV}{dt} = 3000.$$

Using similar triangles: $\frac{w}{h} = \frac{60}{40}$; $w = \frac{3h}{2}$



When $w = 30$: $h = 20$

$$V = \frac{1}{2}wh \times 200 = \frac{1}{2}\left(\frac{3h}{2}\right)h \times 200 = 150h^2$$

$$\frac{dV}{dh} = 300h; \quad \text{sub } h = 20: \quad \frac{dV}{dh} = 6000$$

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 3000 \div 6000 = \boxed{0.5 \text{ cm/s}}$$