

Differentiation Techniques

Basic Differentiation, Chain Rule, Product Rule, Quotient Rule,
Tangents and Normals

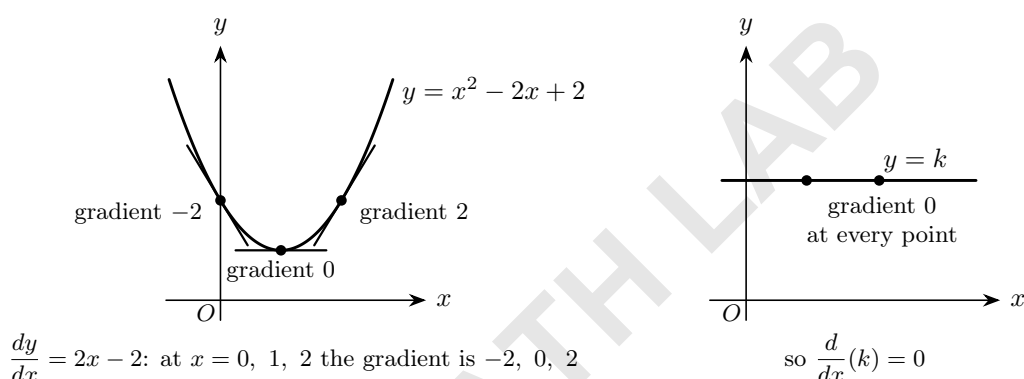
Secondary 4 — G3 Additional Mathematics

Notes

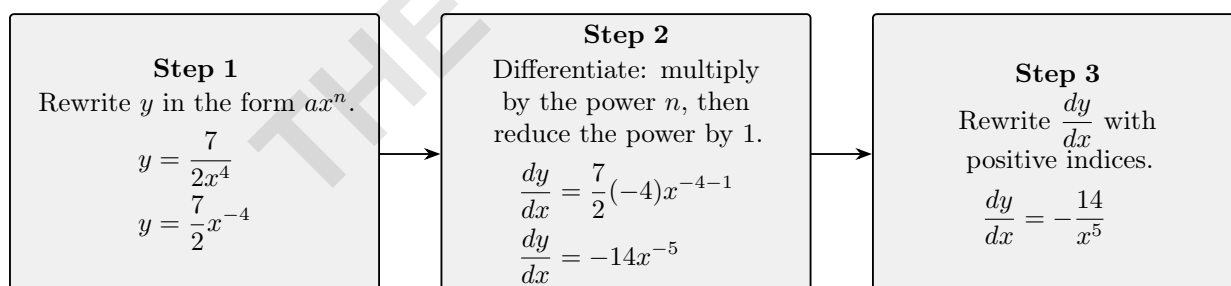
Basic rules

$$\frac{d}{dx}(ax^n) = anx^{n-1} \qquad \frac{d}{dx}(k) = 0$$

The derivative $\frac{dy}{dx}$ is the **gradient function** of the curve: it gives the gradient of the curve at every point. A constant term has a horizontal graph, so its gradient, and hence its derivative, is 0.



Three steps for every term. Before differentiating, write each term as a power of x : a fraction with x in the denominator takes a negative index, and a root takes a fractional index.

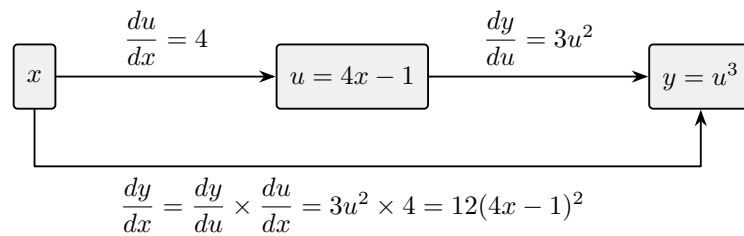


Index form at a glance $\frac{1}{x^n} = x^{-n}$, $\sqrt{x} = x^{0.5}$, $\frac{1}{\sqrt{x}} = x^{-0.5}$, $\sqrt{x^3} = x^{1.5}$, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$,
 $\frac{k}{ax^n} = \frac{k}{a}x^{-n}$.

Chain rule

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x) \qquad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

A bracket raised to a power is a function of a function: $y = (4x - 1)^3$ is $y = u^3$ with $u = 4x - 1$. Differentiate the outer power with the inside left unchanged, then multiply by the derivative of the inside.

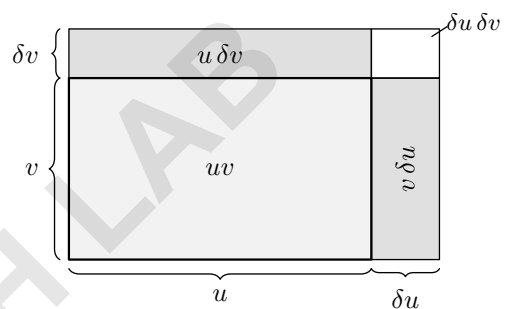


$$\frac{d}{dx} (4x - 1)^3 = \underbrace{3(4x - 1)^2}_{\text{outer: power rule, inside unchanged}} \times \underbrace{(4)}_{\text{inner: derivative of } 4x-1}$$

Product rule

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Picture uv as the area of a rectangle with sides u and v . When x increases by a small amount, u grows by δu and v grows by δv . The area gains the two shaded strips, $u \delta v$ and $v \delta u$, together with a tiny corner $\delta u \delta v$, which is negligible when the changes are small. Dividing the gain by the small change in x gives the product rule: each factor is multiplied by the derivative of the other, and the two products are added.



Before using the rule, write down u , v , $\frac{du}{dx}$ and $\frac{dv}{dx}$ in a table. Then substitute, take out the common factor and simplify.

Quotient rule

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

bottom $\times$ derivative
of top

top $\times$ derivative
of bottom

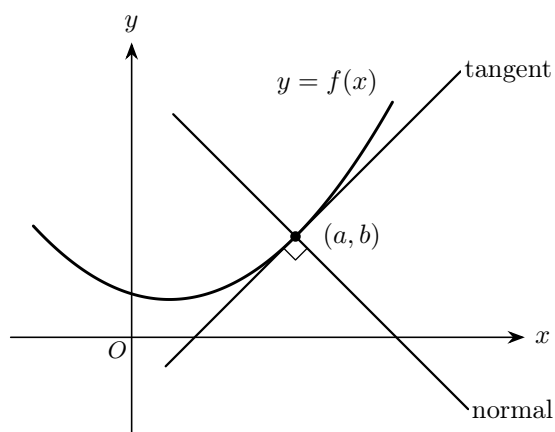
$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{bottom squared}$$

Because of the minus sign the order matters: always begin the numerator with v , the denominator. As with the product rule, tabulate u , v and their derivatives first.

Gradient of a tangent

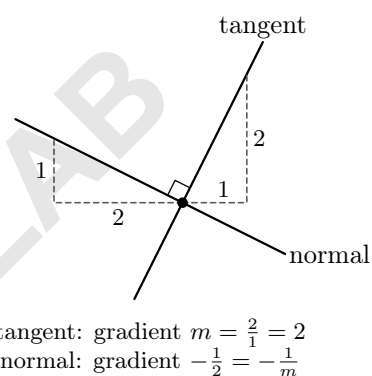
$\frac{dy}{dx}$ represents the **gradient function** of a curve.

When we substitute any x -value into $\frac{dy}{dx}$, we obtain the **gradient** of the **tangent** to the curve at that particular point.



Key concepts

- 1) $\frac{dy}{dx}$ is the gradient of the **tangent** to the curve at a point.
- 2) The **normal** to the curve at a point is the line **perpendicular** to the tangent to the curve at the same point.
- 3) Gradient of normal $= -1 \div \frac{dy}{dx}$ (recall the gradients of perpendicular lines: $m_1 \times m_2 = -1$).
- 4) Both the tangent and the normal are straight lines of the form $y = mx + c$.



Exercises

Section 1: Basic Differentiation

1. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = x^4 - 3x^2$

(b) $y = -2x^3 + 7x^2 - 9$

(c) $y = 3x^5 - \frac{2}{x} + \frac{4}{x^2} - 5x$

(d) $y = 2x^3 + \frac{1}{2x^2} - \frac{5}{3x^3}$

(e) $y = \frac{3\sqrt{x}}{4} + 5x^3$

(f) $y = \frac{5}{3x^2} - 8x + 4x^4 - 1$

(g) $y = 6x^3 - \frac{4}{5x}$

(h) $y = 5x^2 - 2x + 3\sqrt{x}$

2. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \frac{2x^3 - 7}{x^2}$

(b) $y = \frac{x^2 - 4x}{\sqrt{x}}$

(c) $y = \frac{6x^3 + 4x^2 - 3x + 2}{3x}$

(d) $y = (x^2 + 3)^2$

3. Differentiate each function with respect to x .

(a) $y = 5x^4 - 2x^3 + 6x - 1$

(b) $y = \frac{4}{x^5} - \frac{3}{2x}$

(c) $y = 4x^3 - 9x + 2$

(d) $y = 2x - \frac{3}{4x} + \frac{6}{\sqrt{x}}$

(e) $y = 5 - 4x + 6\sqrt{x}$

(f) $y = -\frac{5}{2}x^{-0.6}$

(g) $y = \frac{3}{2}\sqrt[3]{x^4}$

(h) $y = \frac{2}{x^4} + \frac{x^2}{6} - 3$

(i) $y = \frac{4x^3 - 2x + 5}{x}$

(j) $y = \frac{3x^4 - 2x + \sqrt{x}}{x^2}$

(k) $y = \left(x - \frac{3}{x^2}\right)^2$

Section 2: Chain Rule

4. Find the derivative of each expression with respect to x .

(a) $2(3x - 1)^4$

(b) $\frac{3}{4}\left(\frac{x}{2} + 3\right)^6$

(c) $\left(\frac{1}{3}x^3 - 2x\right)^4$

(d) $\sqrt{3x^2 + 6x - 1}$

(e) $\frac{1}{(x^2 - 5x + 1)^3}$

5. Find the derivative of each expression with respect to x .

(a) $\sqrt[3]{2x^2 + 5}$

(b) $4\sqrt{2x^3 + 1}$

(c) $\frac{1}{(1 + 4x - 2x^2)^4}$

(d) $\frac{15}{3 + 5x^2}$

(e) $\frac{5}{2(4 - x^2)}$

6. Find the derivative of each expression with respect to x .

(a) $\frac{4}{9(2 - 3x)^2}$

(b) $\sqrt{x^2 - 4x + 5}$

(c) $(2x^3 - 3x + \pi)^4$

(d) $\frac{1}{(\sqrt{x} + 2)^2}$

Section 3: Product Rule

7. Differentiate each of the following with respect to x , simplifying your answers.
- (a) $(x+2)\sqrt{x-1}$ (b) $x\sqrt{2x+3}$ (c) $2x\sqrt{1-x^3}$ (d) $(x^2-x)(x+3)^5$
8. The equation of a curve is $y = x^3(4-x)^5$. Show that $\frac{dy}{dx}$ can be written as $kx^2(3-2x)(4-x)^4$, where k is a constant, and state the value of k .
9. It is given that $\frac{d}{dx}[(3x+1)\sqrt{x-2}] = \frac{px+q}{2\sqrt{x-2}}$. Find the values of the constants p and q .

Section 4: Quotient Rule

10. Differentiate each of the following with respect to x , simplifying your answers.
- (a) $\frac{x+5}{\sqrt{3x-2}}$ (b) $\frac{4x^2}{\sqrt{2x^2+1}}$ (c) $y = \frac{x^2}{3x+2}$ (d) $y = \frac{x}{\sqrt{x^2+9}}$
11. It is given that $\frac{d}{dx}\left[\frac{2x+5}{x^2+3}\right] = \frac{6+ax+bx^2}{(x^2+3)^2}$. Find the values of the constants a and b .
12. The equation of a curve is $y = \frac{\sqrt{x^2+4}}{2x}$. Show that $\frac{dy}{dx} = \frac{k}{x^2\sqrt{x^2+4}}$, where k is a constant, and state the value of k .

Section 5: Equations of Tangents and Normals

13. The gradient function of a curve is $\frac{dy}{dx} = kx - 2$, where k is a constant. The curve passes through the point $(2, 5)$, and the normal to the curve at the point where $x = 3$ has gradient $-\frac{1}{7}$. Find
- (a) the value of k ,
- (b) the equation of the tangent to the curve at $(2, 5)$.
14. The curve $y = \frac{x^2-9}{2x}$ meets the negative x -axis at the point N . Find the equation of the normal to the curve at N .
15. The line $y = x + k$ is a normal to the curve $y = \frac{4}{(x-2)^2}$. Find the value of k .

Solutions to Exercises

Section 1 — Basic differentiation (Questions 1–3)

$$1(a) \frac{dy}{dx} = \boxed{4x^3 - 6x}$$

$$1(b) \frac{dy}{dx} = \boxed{-6x^2 + 14x}$$

$$1(c) y = 3x^5 - 2x^{-1} + 4x^{-2} - 5x$$

$$\frac{dy}{dx} = 15x^4 + 2x^{-2} - 8x^{-3} - 5 = \boxed{15x^4 + \frac{2}{x^2} - \frac{8}{x^3} - 5}$$

$$1(d) y = 2x^3 + \frac{1}{2}x^{-2} - \frac{5}{3}x^{-3}$$

$$\frac{dy}{dx} = 6x^2 - x^{-3} + 5x^{-4} = \boxed{6x^2 - \frac{1}{x^3} + \frac{5}{x^4}}$$

$$1(e) y = \frac{3}{4}x^{0.5} + 5x^3$$

$$\frac{dy}{dx} = \frac{3}{8}x^{-0.5} + 15x^2 = \boxed{\frac{3}{8\sqrt{x}} + 15x^2}$$

$$1(f) y = \frac{5}{3}x^{-2} - 8x + 4x^4 - 1$$

$$\frac{dy}{dx} = -\frac{10}{3}x^{-3} - 8 + 16x^3 = \boxed{-\frac{10}{3x^3} - 8 + 16x^3}$$

$$1(g) y = 6x^3 - \frac{4}{5}x^{-1}$$

$$\frac{dy}{dx} = 18x^2 + \frac{4}{5}x^{-2} = \boxed{18x^2 + \frac{4}{5x^2}}$$

$$1(h) y = 5x^2 - 2x + 3x^{0.5}$$

$$\frac{dy}{dx} = 10x - 2 + \frac{3}{2}x^{-0.5} = \boxed{10x - 2 + \frac{3}{2\sqrt{x}}}$$

$$2(a) y = \frac{2x^3}{x^2} - \frac{7}{x^2} = 2x - 7x^{-2} \text{ [divide each term by } x^2]$$

$$\frac{dy}{dx} = 2 + 14x^{-3} = \boxed{2 + \frac{14}{x^3}}$$

$$2(b) y = \frac{x^2 - 4x}{x^{0.5}} = x^{1.5} - 4x^{0.5}$$

$$\frac{dy}{dx} = 1.5x^{0.5} - 2x^{-0.5} = \boxed{\frac{3}{2}\sqrt{x} - \frac{2}{\sqrt{x}}}$$

$$2(c) y = 2x^2 + \frac{4}{3}x - 1 + \frac{2}{3}x^{-1}$$

$$\frac{dy}{dx} = 4x + \frac{4}{3} - \frac{2}{3}x^{-2} = \boxed{4x + \frac{4}{3} - \frac{2}{3x^2}}$$

$$2(d) y = (x^2 + 3)^2 = x^4 + 6x^2 + 9 \text{ [expand first]}$$

$$\frac{dy}{dx} = \boxed{4x^3 + 12x}$$

$$3(a) \frac{dy}{dx} = \boxed{20x^3 - 6x^2 + 6}$$

$$3(b) y = 4x^{-5} - \frac{3}{2}x^{-1}$$

$$\frac{dy}{dx} = -20x^{-6} + \frac{3}{2}x^{-2} = \boxed{-\frac{20}{x^6} + \frac{3}{2x^2}}$$

$$3(c) \frac{dy}{dx} = \boxed{12x^2 - 9}$$

$$3(d) y = 2x - \frac{3}{4}x^{-1} + 6x^{-0.5}$$

$$\frac{dy}{dx} = 2 + \frac{3}{4}x^{-2} - 3x^{-1.5} = \boxed{2 + \frac{3}{4x^2} - \frac{3}{\sqrt{x^3}}}$$

$$3(e) y = 5 - 4x + 6x^{0.5}$$

$$\frac{dy}{dx} = -4 + 3x^{-0.5} = \boxed{-4 + \frac{3}{\sqrt{x}}}$$

$$3(f) \frac{dy}{dx} = -\frac{5}{2}(-0.6)x^{-1.6} = 1.5x^{-1.6} = \boxed{\frac{3}{2x^{1.6}}}$$

$$3(g) y = \frac{3}{2}x^{\frac{4}{3}}$$

$$\frac{dy}{dx} = \frac{3}{2} \cdot \frac{4}{3}x^{\frac{1}{3}} = 2x^{\frac{1}{3}} = \boxed{2\sqrt[3]{x}}$$

$$3(h) y = 2x^{-4} + \frac{1}{6}x^2 - 3$$

$$\frac{dy}{dx} = -8x^{-5} + \frac{1}{3}x = \boxed{-\frac{8}{x^5} + \frac{x}{3}}$$

$$3(i) y = 4x^2 - 2 + 5x^{-1}$$

$$\frac{dy}{dx} = 8x - 5x^{-2} = \boxed{8x - \frac{5}{x^2}}$$

$$3(j) y = 3x^2 - 2x^{-1} + x^{-1.5}$$

$$\frac{dy}{dx} = 6x + 2x^{-2} - 1.5x^{-2.5} = \boxed{6x + \frac{2}{x^2} - \frac{3}{2\sqrt{x^5}}}$$

$$3(k) y = x^2 - 6x^{-1} + 9x^{-4} \text{ [expand first]}$$

$$\frac{dy}{dx} = 2x + 6x^{-2} - 36x^{-5} = \boxed{2x + \frac{6}{x^2} - \frac{36}{x^5}}$$

Section 2 — Chain rule (Questions 4–6)

$$4(a) 2(4)(3x - 1)^3(3) = \boxed{24(3x - 1)^3}$$

$$4(b) \frac{3}{4}(6)\left(\frac{x}{2} + 3\right)^5\left(\frac{1}{2}\right) = \boxed{\frac{9}{4}\left(\frac{x}{2} + 3\right)^5}$$

$$4(c)$$

$$4\left(\frac{1}{3}x^3 - 2x\right)^3(x^2 - 2) = \boxed{4(x^2 - 2)\left(\frac{1}{3}x^3 - 2x\right)^3}$$

4(d)

$$\frac{d}{dx} (3x^2 + 6x - 1)^{0.5} = \frac{1}{2} (3x^2 + 6x - 1)^{-0.5} (6x + 6)$$

$$= \frac{3x + 3}{\sqrt{3x^2 + 6x - 1}} = \boxed{\frac{3(x + 1)}{\sqrt{3x^2 + 6x - 1}}}$$

4(e)

$$\frac{d}{dx} (x^2 - 5x + 1)^{-3} = -3 (x^2 - 5x + 1)^{-4} (2x - 5)$$

$$= \boxed{\frac{-3(2x - 5)}{(x^2 - 5x + 1)^4}}$$

$$\mathbf{5(a)} \quad \frac{d}{dx} (2x^2 + 5)^{\frac{1}{3}} = \frac{1}{3} (2x^2 + 5)^{-\frac{2}{3}} (4x)$$

$$= \boxed{\frac{4x}{3\sqrt[3]{(2x^2 + 5)^2}}}$$

$$\mathbf{5(b)} \quad \frac{d}{dx} 4(2x^3 + 1)^{0.5} = 4 \cdot \frac{1}{2} (2x^3 + 1)^{-0.5} (6x^2)$$

$$= \boxed{\frac{12x^2}{\sqrt{2x^3 + 1}}}$$

5(c)

$$\frac{d}{dx} (1 + 4x - 2x^2)^{-4} = -4 (1 + 4x - 2x^2)^{-5} (4 - 4x)$$

$$= \boxed{\frac{16(x - 1)}{(1 + 4x - 2x^2)^5}}$$

$$\mathbf{5(d)} \quad \frac{d}{dx} 15(3 + 5x^2)^{-1} = -15(3 + 5x^2)^{-2} (10x)$$

$$= \boxed{-\frac{150x}{(3 + 5x^2)^2}}$$

$$\mathbf{5(e)} \quad \frac{d}{dx} \frac{5}{2} (4 - x^2)^{-1} = -\frac{5}{2} (4 - x^2)^{-2} (-2x)$$

$$= \boxed{\frac{5x}{(4 - x^2)^2}}$$

$$\mathbf{6(a)} \quad \frac{d}{dx} \frac{4}{9} (2 - 3x)^{-2} = \frac{4}{9} (-2) (2 - 3x)^{-3} (-3)$$

$$= \frac{24}{9(2 - 3x)^3} = \boxed{\frac{8}{3(2 - 3x)^3}}$$

$$\mathbf{6(b)} \quad \frac{1}{2} (x^2 - 4x + 5)^{-0.5} (2x - 4) = \boxed{\frac{x - 2}{\sqrt{x^2 - 4x + 5}}}$$

6(c) [π is a constant]

$$4(2x^3 - 3x + \pi)^3 (6x^2 - 3) = \boxed{12(2x^2 - 1)(2x^3 - 3x + \pi)^3}$$

$$\mathbf{6(d)} \quad \frac{d}{dx} (x^{0.5} + 2)^{-2} = -2 (x^{0.5} + 2)^{-3} (\frac{1}{2}x^{-0.5})$$

$$= \boxed{-\frac{1}{\sqrt{x}(\sqrt{x} + 2)^3}}$$

Section 3 — Product rule (Questions 7–9)

$$\mathbf{7(a)} \quad \begin{array}{l|l} u = x + 2 & v = (x - 1)^{0.5} \\ \hline \frac{du}{dx} = 1 & \frac{dv}{dx} = \frac{1}{2}(x - 1)^{-0.5} \end{array}$$

$$\frac{dy}{dx} = (x - 1)^{0.5}(1) + (x + 2) \cdot \frac{1}{2}(x - 1)^{-0.5}$$

$$= \frac{1}{2}(x - 1)^{-0.5} [2(x - 1) + (x + 2)] = \boxed{\frac{3x}{2\sqrt{x - 1}}}$$

$$\mathbf{7(b)} \quad \begin{array}{l|l} u = x & v = (2x + 3)^{0.5} \\ \hline \frac{du}{dx} = 1 & \frac{dv}{dx} = \frac{1}{2}(2x + 3)^{-0.5}(2) \end{array}$$

$$\frac{dy}{dx} = (2x + 3)^{0.5}(1) + x(2x + 3)^{-0.5}$$

$$= (2x + 3)^{-0.5} [(2x + 3) + x] = \boxed{\frac{3(x + 1)}{\sqrt{2x + 3}}}$$

$$\mathbf{7(c)} \quad \begin{array}{l|l} u = 2x & v = (1 - x^3)^{0.5} \\ \hline \frac{du}{dx} = 2 & \frac{dv}{dx} = \frac{1}{2}(1 - x^3)^{-0.5}(-3x^2) \end{array}$$

$$\frac{dy}{dx} = (1 - x^3)^{0.5}(2) + 2x \cdot \frac{1}{2}(1 - x^3)^{-0.5}(-3x^2)$$

$$= (1 - x^3)^{-0.5} [2(1 - x^3) - 3x^3] = \boxed{\frac{2 - 5x^3}{\sqrt{1 - x^3}}}$$

$$\mathbf{7(d)} \quad \begin{array}{l|l} u = x^2 - x & v = (x + 3)^5 \\ \hline \frac{du}{dx} = 2x - 1 & \frac{dv}{dx} = 5(x + 3)^4 \end{array}$$

$$\frac{dy}{dx} = (x + 3)^5(2x - 1) + (x^2 - x) \cdot 5(x + 3)^4$$

$$= (x + 3)^4 [(x + 3)(2x - 1) + 5x^2 - 5x]$$

$$= (x + 3)^4 [2x^2 + 5x - 3 + 5x^2 - 5x] =$$

$$\boxed{(x + 3)^4(7x^2 - 3)}$$

$$\mathbf{8} \quad \begin{array}{l|l} u = x^3 & v = (4 - x)^5 \\ \hline \frac{du}{dx} = 3x^2 & \frac{dv}{dx} = 5(4 - x)^4(-1) \end{array}$$

$$\frac{dy}{dx} = (4 - x)^5(3x^2) + x^3(-5)(4 - x)^4$$

$$= x^2(4 - x)^4 [3(4 - x) - 5x] = x^2(4 - x)^4(12 - 8x)$$

$$= 4x^2(3 - 2x)(4 - x)^4, \quad \boxed{k = 4}$$

$$\mathbf{9} \quad \begin{array}{l|l} u = 3x + 1 & v = (x - 2)^{0.5} \\ \hline \frac{du}{dx} = 3 & \frac{dv}{dx} = \frac{1}{2}(x - 2)^{-0.5} \end{array}$$

$$\frac{d}{dx} [(3x + 1)\sqrt{x - 2}]$$

$$= 3(x - 2)^{0.5} + (3x + 1) \cdot \frac{1}{2}(x - 2)^{-0.5}$$

$$= \frac{1}{2}(x - 2)^{-0.5} [6(x - 2) + 3x + 1] = \frac{9x - 11}{2\sqrt{x - 2}}$$

$$\boxed{p = 9, q = -11}$$

Section 4 — Quotient rule (Questions 10–12)

$$\begin{array}{l}
 \text{10(a)} \quad \left. \begin{array}{l} u = x + 5 \\ \frac{du}{dx} = 1 \end{array} \right| \left. \begin{array}{l} v = (3x - 2)^{0.5} \\ \frac{dv}{dx} = \frac{1}{2}(3x - 2)^{-0.5}(3) \end{array} \right. \\
 \frac{dy}{dx} = \frac{(3x - 2)^{0.5}(1) - (x + 5) \cdot \frac{3}{2}(3x - 2)^{-0.5}}{3x - 2} \\
 = \frac{\frac{1}{2}(3x - 2)^{-0.5}[2(3x - 2) - 3(x + 5)]}{3x - 2} =
 \end{array}$$

$$\boxed{\frac{3x - 19}{2\sqrt{(3x - 2)^3}}}$$

$$\begin{array}{l}
 \text{10(b)} \quad \left. \begin{array}{l} u = 4x^2 \\ \frac{du}{dx} = 8x \end{array} \right| \left. \begin{array}{l} v = (2x^2 + 1)^{0.5} \\ \frac{dv}{dx} = \frac{1}{2}(2x^2 + 1)^{-0.5}(4x) \end{array} \right. \\
 \frac{dy}{dx} = \frac{(2x^2 + 1)^{0.5}(8x) - 4x^2 \cdot 2x(2x^2 + 1)^{-0.5}}{2x^2 + 1} \\
 = \frac{(2x^2 + 1)^{-0.5}[8x(2x^2 + 1) - 8x^3]}{2x^2 + 1} = \frac{8x^3 + 8x}{(2x^2 + 1)^{1.5}}
 \end{array}$$

$$= \boxed{\frac{8x(x^2 + 1)}{\sqrt{(2x^2 + 1)^3}}}$$

$$\begin{array}{l}
 \text{10(c)} \quad \left. \begin{array}{l} u = x^2 \\ \frac{du}{dx} = 2x \end{array} \right| \left. \begin{array}{l} v = 3x + 2 \\ \frac{dv}{dx} = 3 \end{array} \right. \\
 \frac{dy}{dx} = \frac{(3x + 2)(2x) - x^2(3)}{(3x + 2)^2} = \frac{3x^2 + 4x}{(3x + 2)^2} = \boxed{\frac{x(3x + 4)}{(3x + 2)^2}}
 \end{array}$$

$$\begin{array}{l}
 \text{10(d)} \quad \left. \begin{array}{l} u = x \\ \frac{du}{dx} = 1 \end{array} \right| \left. \begin{array}{l} v = (x^2 + 9)^{0.5} \\ \frac{dv}{dx} = \frac{1}{2}(x^2 + 9)^{-0.5}(2x) \end{array} \right. \\
 \frac{dy}{dx} = \frac{(x^2 + 9)^{0.5}(1) - x \cdot x(x^2 + 9)^{-0.5}}{x^2 + 9} \\
 = \frac{(x^2 + 9)^{-0.5}[(x^2 + 9) - x^2]}{x^2 + 9} = \boxed{\frac{9}{\sqrt{(x^2 + 9)^3}}}
 \end{array}$$

$$\begin{array}{l}
 \text{11} \quad \left. \begin{array}{l} u = 2x + 5 \\ \frac{du}{dx} = 2 \end{array} \right| \left. \begin{array}{l} v = x^2 + 3 \\ \frac{dv}{dx} = 2x \end{array} \right. \\
 \frac{d}{dx} \left[\frac{2x + 5}{x^2 + 3} \right] = \frac{(x^2 + 3)(2) - (2x + 5)(2x)}{(x^2 + 3)^2} \\
 = \frac{2x^2 + 6 - 4x^2 - 10x}{(x^2 + 3)^2} = \frac{6 - 10x - 2x^2}{(x^2 + 3)^2}
 \end{array}$$

$$\boxed{a = -10, b = -2}$$

$$\begin{array}{l}
 \text{12} \quad \left. \begin{array}{l} u = (x^2 + 4)^{0.5} \\ \frac{du}{dx} = \frac{1}{2}(x^2 + 4)^{-0.5}(2x) \end{array} \right| \left. \begin{array}{l} v = 2x \\ \frac{dv}{dx} = 2 \end{array} \right. \\
 \frac{dy}{dx} = \frac{2x \cdot x(x^2 + 4)^{-0.5} - (x^2 + 4)^{0.5}(2)}{(2x)^2} \\
 = \frac{(x^2 + 4)^{-0.5}[2x^2 - 2(x^2 + 4)]}{4x^2} = \frac{-8}{4x^2\sqrt{x^2 + 4}} \\
 = \frac{-2}{x^2\sqrt{x^2 + 4}}, \quad \boxed{k = -2}
 \end{array}$$

Section 5 — Tangents and normals (Questions 13–15)

$$\text{13(a)} \text{ At } x = 3, \text{ gradient of tangent} = -1 \div \left(-\frac{1}{7}\right) = 7$$

$$3k - 2 = 7, \quad \boxed{k = 3}$$

$$\text{13(b)} \text{ At } (2, 5): \frac{dy}{dx} = 3(2) - 2 = 4$$

$$y = 4x + c; \text{ sub } (2, 5): c = -3. \quad \boxed{y = 4x - 3}$$

$$\text{14 Sub } y = 0: x^2 - 9 = 0, x = \pm 3; \text{ on the negative } x\text{-axis, } x = -3, \text{ so } N(-3, 0).$$

$$y = \frac{1}{2}x - \frac{9}{2}x^{-1}; \quad \frac{dy}{dx} = \frac{1}{2} + \frac{9}{2}x^{-2}$$

$$\text{When } x = -3, \frac{dy}{dx} = \frac{1}{2} + \frac{1}{2} = 1; \text{ gradient of normal} = -1$$

$$y = -x + c; \text{ sub } (-3, 0): c = -3. \quad \boxed{y = -x - 3}$$

$$\text{15 } y = 4(x - 2)^{-2}; \quad \frac{dy}{dx} = -8(x - 2)^{-3} = \frac{-8}{(x - 2)^3}$$

$$y = x + k \text{ has gradient } 1. \text{ The normal has gradient } 1, \text{ so the tangent has gradient } -1 \div 1 = -1.$$

$$\frac{-8}{(x - 2)^3} = -1; (x - 2)^3 = 8; x - 2 = 2; x = 4$$

$$y = \frac{4}{(4 - 2)^2} = 1$$

$$\text{Sub } (4, 1) \text{ into } y = x + k: 1 = 4 + k, \quad \boxed{k = -3}$$