Quadratic Equations and Functions

Completing the Square, Solving, Word Problems and Graph Sketching

Secondary 3 — G3 Mathematics

Notes

Completing the Square Formula

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

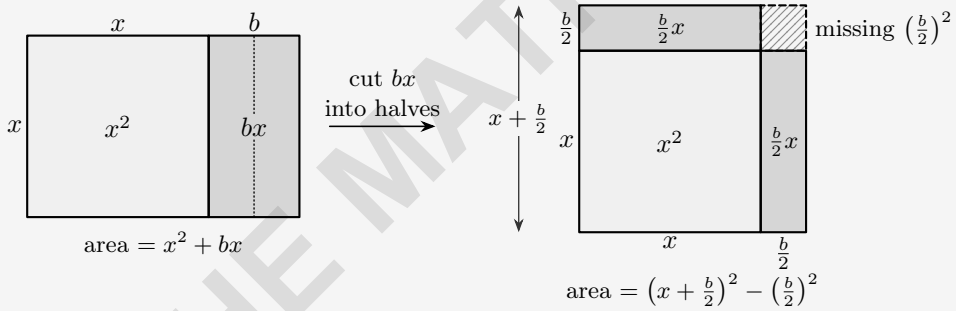
Example:

$$x^2 + 7x = \left(x + \tfrac{7}{2}\right)^2 - \left(\tfrac{7}{2}\right)^2$$
$$= \left(x + \tfrac{7}{2}\right)^2 - 12.25$$

$$x^2 - 8x = (x - 4)^2 - (-4)^2$$
$$= (x - 4)^2 - 16$$

How to Remember the Formula? $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

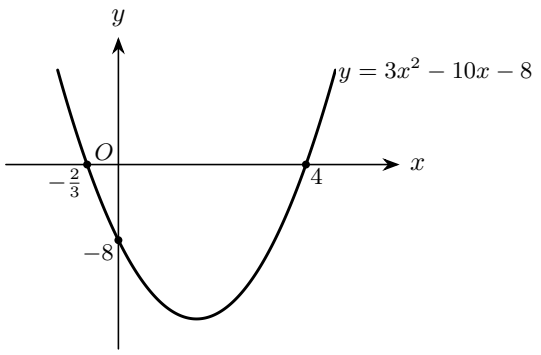
Think of $x^2 + bx$ as an area: an x by x square with an x by b strip beside it. Cut the strip into two halves of width $\frac{b}{2}$ and move one half to the top. The pieces now almost fill a square of side $x + \frac{b}{2}$; only the small corner square of area $\left(\frac{b}{2}\right)^2$ is missing.



So: *halve* the coefficient of x, *square* the bracket, then *subtract* the square of the half.

When the coefficient of x^2 is -1, take -1 out of the x^2 and x terms first, then complete the square inside the bracket as above.

What the solutions mean on a graph. The solutions of $ax^2 + bx + c = 0$ are the x-coordinates of the points where the graph of $y = ax^2 + bx + c$ meets the x-axis. The graph of $y = 3x^2 - 10x - 8$ crosses the x-axis at $x = -\frac{2}{3}$ and $x = 4$. The notes return to these graphs later on.



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Solving $x^2 + bx + c = 0$ by completing the square

1. Bring every term to the left-hand side, so that the equation reads $x^2 + bx + c = 0$.
2. Complete the square on the left, using the completing-the-square formula: $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0$.
3. Move the constant terms to the right-hand side and combine them into a single number, keeping fractions exact: $\left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$.
4. Take the square root of both sides, writing $\pm$: $x + \frac{b}{2} = \pm\sqrt{\left(\frac{b}{2}\right)^2 - c}$.
5. Make x the subject, $x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$, then work out the "+" value and the "−" value separately, rounding only the final answers when they are not exact.

When a part begins with **"Hence"**, start from the completed square found in the part before it.

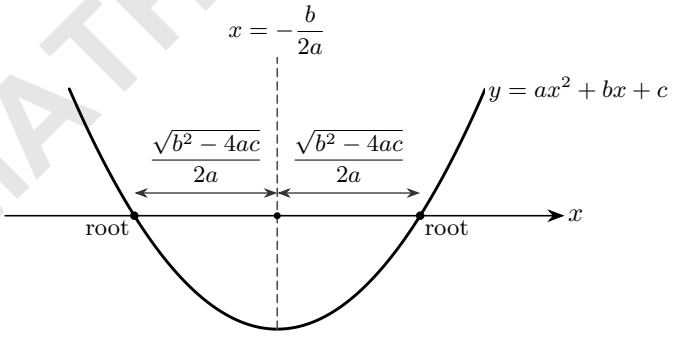
Never drop the $\pm$: a positive number has two square roots, and each one gives a solution. If the right-hand side in step 3 is negative, the equation has **no real solutions**, since a square is never negative; if it is 0, there is only one solution, $x = -\frac{b}{2}$.

The general formula

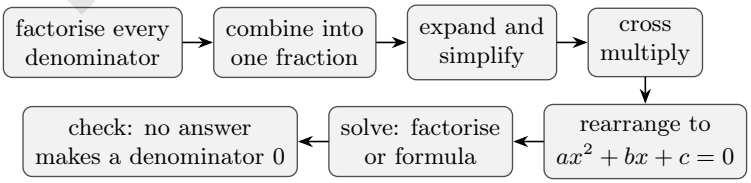
To solve a quadratic equation in the form $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The formula starts at the line of symmetry $x = -\frac{b}{2a}$ of the graph of $y = ax^2 + bx + c$, then steps the same distance $\frac{\sqrt{b^2-4ac}}{2a}$ to the left and to the right to reach the two roots.



The steps for solving a fractional equation, in order:



Recap

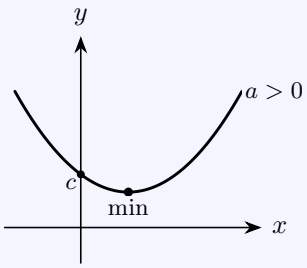
The equation of a quadratic curve is $\boldsymbol{y = ax^2 + bx + c}$. a is the coefficient of x^2; c is the y-intercept of the graph.

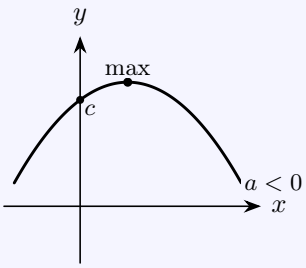
When coefficient of x^2 is positive (i.e. $a > 0$): the graph is U-shaped and it will have a **minimum point**.

When coefficient of x^2 is negative (i.e. $a < 0$): the graph is n-shaped and it will have a **maximum point**.

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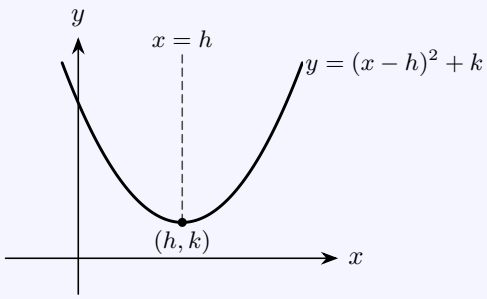
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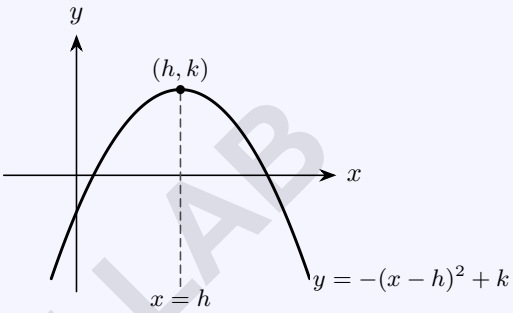




Sketching $y = \pm(x-h)^2 + k$

Complete the square of the quadratic equation into $\boldsymbol{y = \pm(x-h)^2 + k}$. The **turning point** (max or min) is $\boldsymbol{(h, k)}$.

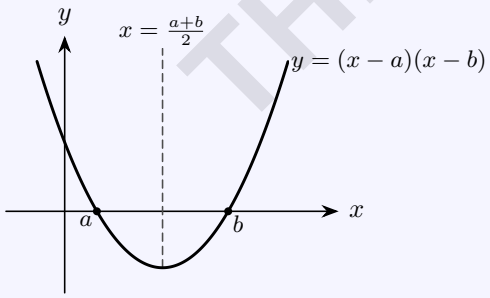


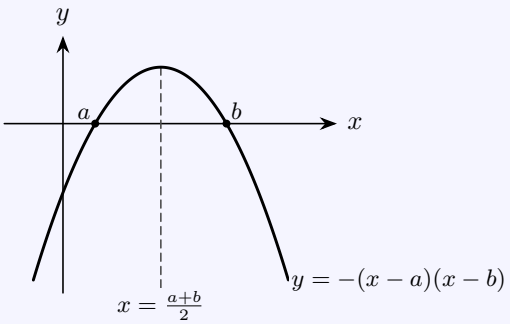


- To get x-intercepts, sub $y = 0$.
- To get the y-intercept, sub $x = 0$.
- The line of symmetry is $x = h$.

Sketching $y = \pm(x-a)(x-b)$

Factorise the quadratic equation into $\boldsymbol{y = \pm(x-a)(x-b)}$. To find x-intercepts, sub $y = 0$: $\therefore x = a$ and $x = b$, so $\boldsymbol{(a, 0)}$ and $\boldsymbol{(b, 0)}$ are the x-intercepts.





- To obtain the y-intercept, sub $x = 0$.
- Line of symmetry: $x = \dfrac{a+b}{2}$, halfway between the two x-intercepts.
- The line of symmetry also gives the x-coordinate of the max/min point.
- To obtain the y-coordinate of the max/min point, sub $x = \dfrac{a+b}{2}$ into the equation.

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Finding $y = ax^2 + bx + c$ from a sketch

1. **A missing x-intercept.** The line of symmetry $x = h$ lies halfway between the two x-intercepts p and q, so $\dfrac{p+q}{2} = h$; solve this for the unknown intercept. The line of symmetry passes through the turning point, so a given turning point also gives h.
2. **Choose the form** that uses what the sketch gives:
   - both x-intercepts p and q: $y = a(x-p)(x-q)$;
   - the turning point (h, k): $y = a(x-h)^2 + k$.
3. **Find a.** Substitute the coordinates of one more point on the graph, often the point where it cuts the y-axis, and solve for a.
4. **Expand** and simplify into the form $y = ax^2 + bx + c$, then read off a, b and c.

To find a, substitute a point that is not already built into the form: in $y = a(x-p)(x-q)$ an x-intercept makes both sides 0, and in $y = a(x-h)^2 + k$ the turning point only gives $k = k$; neither says anything about a. Watch the signs: an intercept at $x = -m$ gives the factor $(x+m)$. Check the answer: $a > 0$ for a U-shaped graph, $a < 0$ for an n-shaped one, and c is the y-intercept.

Solving a quadratic equation graphically

1. **Table of values.** Substitute each given value of x into the equation to find y. If the question gives no table, make your own, using each whole-number value of x in the range given.
2. **Plot and join.** Using the scales given in the question, plot each point (x, y) and join the points with one *smooth curve*, not with straight line segments. Near the turning point the curve is rounded, not pointed.
3. **Solve "plotted expression $= 0$".** The solutions are the x-coordinates of the points where the curve meets the x-axis (the line $y = 0$).
4. **Solve another equation with a straight line.** Rearrange the equation so that one side is *exactly* the plotted expression and the other side has no x^2 term, for example $x^2 - x - 3 = -x$ (multiply through first if the x^2 terms do not match). Draw the straight line, here $y = -x$, on the same axes, using three points and a ruler. The solutions are the x-coordinates of the points where the line meets the curve.

Why it works: where the line meets the curve, both have the same y-value, so there $x^2 - x - 3 = -x$, which is the rearranged equation.

Number of solutions = number of meeting points: two points give two solutions; a curve that only touches the x-axis gives one solution; if they do not meet, there are no real solutions.

Multiply *every* term when matching the x^2 terms. A common slip is to multiply only x^2 and forget the other terms, which gives the wrong line.

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Exercises

Unless a question says otherwise, give non-exact answers correct to 3 significant figures.

Section 1: Completing the Square

1. Write each expression in the form $(x+a)^2+b$.

(a) $x^2+10x+31$ (b) x^2-7x-2

2. Write each expression in the form $(x+a)^2+b$.

(a) $x^2-1.4x+0.3$ (b) $x^2+\frac{2}{3}x+\frac{1}{9}$

3. (a) Write $x^2-12x+11$ in the form $(x-a)^2-b$, where a and b are constants.
(b) Write $x^2-\frac{7}{2}x+1$ in the form $(x-a)^2+b$.
(c) Write $(x-7)(x+3)$ in the form $(x-a)^2+b$.

Section 2: When the Coefficient of x^2 is -1

4. By completing the square, write $-x^2+10x-7$ in the form $-(x-h)^2+k$.

5. By completing the square, write $-x^2-3x+2$ in the form $-(x-h)^2+k$.

Section 3: Solving through Factorisation

6. Solve each equation by factorisation.

(a) $4x^2=10x$ (b) $2x^2=7x+15$

7. Solve each equation by factorisation.

(a) $49-9x^2=0$ (b) $(3x-1)^2=16$

8. Solve each equation by factorisation.

(a) $x^2+\frac{5}{4}x=\frac{3}{2}$ (b) $(x+2)(x-3)=14$
(c) $x(2x+1)=3(2x+1)$

Section 4: Solving through Completing the Square

9. (a) (i) Complete the square for x^2+9x+5.
(ii) Hence solve $x^2+9x+5=0$, giving your answers correct to 3 s.f.
(b) By completing the square, solve $x^2-6x+4=0$, giving your answers correct to 3 s.f.
(c) By completing the square, solve $x^2-\frac{5}{2}x-4=0$, giving your answers correct to 2 d.p.

10. (a) By completing the square, solve $x^2+4x-9=0$, giving your answers correct to 2 d.p.
(b) By completing the square, solve $x^2-3x-7=0$, giving your answers correct to 2 d.p.

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11. (i) Complete the square for $x^2 + \frac{2}{5}x - 6$.

(ii) Hence solve $x^2 + \frac{2}{5}x - 6 = 0$, giving your answers correct to 3 s.f.

Section 5: Solving through the Quadratic Formula

12. Use the quadratic formula to solve each equation, giving your answers correct to 2 decimal places. (Show the formula with the values substituted.)

(a) $x^2 + 5x + 1 = 0$ (b) $5x^2 - 11x - 6 = 0$

13. Use the quadratic formula to solve each equation, giving your answers correct to 2 decimal places. (Show the formula with the values substituted.)

(a) $x^2 - 4x = 11$ (b) $\dfrac{x}{3} = \dfrac{2x+1}{4x}$

14. Use the general formula to solve each equation, giving your answers correct to 3 s.f.:

(a) $-3x^2 + 4x + 9 = 0$ (b) $3x^2 - 12x + 6 = 0$ (c) $-5x + 3x^2 - 28 = 0$

Section 6: Solving Fractional Equations

15. Use the general formula to solve each equation:

(a) $2x + \dfrac{15}{x} = 11$

(b) $\dfrac{2}{x} + \dfrac{3}{x+3} = \dfrac{2}{3}$

(c) $\dfrac{1}{x+2} - \dfrac{1}{3x+1} = \dfrac{1}{10}$

16. Use the general formula to solve each equation, giving non-exact answers correct to 3 s.f.:

(a) $\dfrac{2}{x^2-9} + \dfrac{1}{x-3} = 3$

(b) $\dfrac{7x+5}{3x^2-5x-2} - \dfrac{2}{x-2} = 1$

(c) $\dfrac{5}{3x-2} + \dfrac{x^2}{2-3x} = \dfrac{3}{2}$

17. Use the general formula to solve each equation, giving non-exact answers correct to 3 s.f.:

(a) $\dfrac{x+2}{4} - \dfrac{3}{x-1} = 2$

(b) $\dfrac{2m}{m+2} - \dfrac{3}{m^2-4} = 1$

(c) $\dfrac{3x}{x+2} - \dfrac{x-1}{x} = \dfrac{5}{x}$

Section 7: Word Problems

18. A courier drives a van 80 km along an expressway at x km/h. She then drives a further 24 km through town at $(x - 45)$ km/h. The whole journey takes 2 hours.

(i) Write down, in terms of x, the time taken on the expressway.

(ii) Write down, in terms of x, the time taken to drive through town.

(iii) Form an equation in x and show that it reduces to $x^2 - 97x + 1800 = 0$.

(iv) Solve the equation and find the time the courier took to drive through town, giving your answer in minutes.

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19. A delivery drone flies 30 km from a depot to a farm at a constant speed of x km/h without any load. On the way back from the farm to the depot it carries a parcel, and its speed is 8 km/h lower.

(a) Write down, in terms of x, the time taken for the flight from the depot to the farm.

(b) Write down, in terms of x, the time taken for the flight back to the depot.

(c) Given that the flight back takes 10 minutes longer than the flight out, form an equation in x and show that it simplifies to $x^2 - 8x - 1440 = 0$.

(d) Solve the equation $x^2 - 8x - 1440 = 0$, giving your answers to 2 decimal places.

(e) Find the time taken for the flight back to the depot, giving your answer to the nearest minute.

20. An express bus and a regular bus each travel 240 km between two cities. The express bus travels at an average speed of x km/h, and the regular bus at an average speed that is 20 km/h slower. The regular bus takes 1 hour 30 minutes longer than the express bus for the journey. Form an equation in x and show that it reduces to $x^2 - 20x - 3200 = 0$. Solve this equation and hence find the average speed of the express bus.

21. A tap fills an empty water tank in x hours. An outlet at the bottom of the tank empties a full tank in $(x + 3)$ hours. When the tap and the outlet are both left open, the empty tank is filled in 12 hours. Form an equation in x and show that it simplifies to $x^2 + 3x - 36 = 0$. Solve the equation and find the time the outlet alone takes to empty a full tank.

22. A new printing machine prints a batch of brochures in x hours.

(i) Express, in terms of x, the fraction of the batch printed when the new machine runs for one hour.

(ii) An older machine on its own takes four hours longer than the new machine to print the same batch. Express, in terms of x, the fraction of the batch printed when the older machine runs alone for one hour.

(iii) Given that the two machines working together print the batch in 5 hours, form an equation in x and show that it can be simplified to $x^2 - 6x - 20 = 0$.

(iv) Solve the equation, giving the solutions correct to two decimal places.

(v) Calculate the percentage of the batch that is printed if the new machine alone runs for 2 hours, giving your answer correct to one decimal place.

23. At a charity drive, Siti packs 48 gift boxes in x minutes, while Jun takes $(x + 12)$ minutes to pack 30 gift boxes. Working together at these rates, they pack a total of 102 gift boxes in 36 minutes.

(a) Form an equation in x and show that it reduces to $17x^2 - 264x - 3456 = 0$.

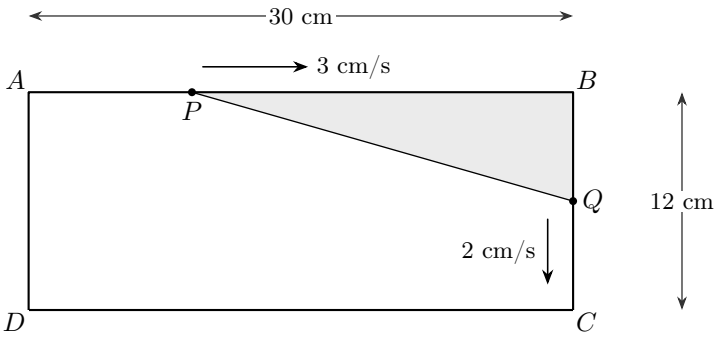
(b) Solve the equation $17x^2 - 264x - 3456 = 0$.

(c) Hence, find the difference between the numbers of gift boxes packed by Siti and by Jun in the 36 minutes.

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24. $ABCD$ is a rectangle with $AB = 30$ cm and $BC = 12$ cm. A point P starts at A and moves along AB towards B at a constant speed of 3 cm/s. At the same time, a point Q starts at B and moves along BC towards C at a constant speed of 2 cm/s.



(a) Write down, in terms of t, the length of PB after t seconds (for $0 < t \leq 6$).

(b) Write down, in terms of t, the length of BQ after t seconds (for $0 < t \leq 6$).

Given that, t seconds after the start (for $0 < t \leq 6$), the area of triangle PBQ is 60 cm^2,

(c) form an equation in t and show that it reduces to $t^2 - 10t + 20 = 0$,

(d) solve the equation, explain why only one of the solutions is valid, and hence find the length of PQ at that instant.

25. A group of x friends book a chalet for a weekend for \$1350 and agree to share the cost equally. Three of them later find that they cannot go, so the remaining friends share the \$1350 equally, and each of them pays \$75 more than originally planned.

(a) Use this information to write down an equation involving x and show that it simplifies to $x^2 - 3x - 54 = 0$.

(b) Solve the equation $x^2 - 3x - 54 = 0$ and find the amount paid by each friend who went to the chalet.

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Section 8: Sketching $y = \pm(x-h)^2 + k$

26. For each graph below, draw a sketch and state the equation of the line of symmetry, the coordinates of the maximum or minimum point, and the x- and y-intercepts where they exist.

(a) $y = (x+1)^2 + 4$ (b) $y = -(x-4)^2 + 3$ (c) $y = x^2 + 2$

27. (i) Write $x^2 + 8x + 7$ in the form $(x+a)^2 + b$.

(ii) Hence sketch the graph of $y = x^2 + 8x + 7$.

28. (a) Express $x^2 - 12x + 40$ in the form $(x-a)^2 + b$.

(b) Write down the coordinates of the turning point of the graph of $y = x^2 - 12x + 40$.

(c) Explain why the equation $x^2 - 12x + 40 = 0$ has no real solutions.

Section 9: Sketching $y = \pm(x-a)(x-b)$

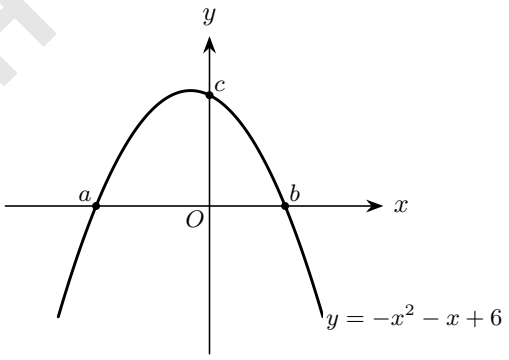
29. Sketch each of these graphs, stating the equation of the line of symmetry, the coordinates of the maximum or minimum point, and the x- and y-intercepts.

(a) $y = (x+4)(x-1)$

(b) $y = x(x+6)$

(c) $y = -(x-1)(x-5)$

30. The diagram shows the graph of $y = -x^2 - x + 6$. The graph cuts the x-axis at $(a, 0)$ and $(b, 0)$, where $a < b$, and the y-axis at $(0, c)$. Find the values of a, b and c.



31. Write each equation in the form $y = \pm(x-a)(x-b)$. Then sketch the graph, stating the line of symmetry, the maximum or minimum point, and the x- and y-intercepts.

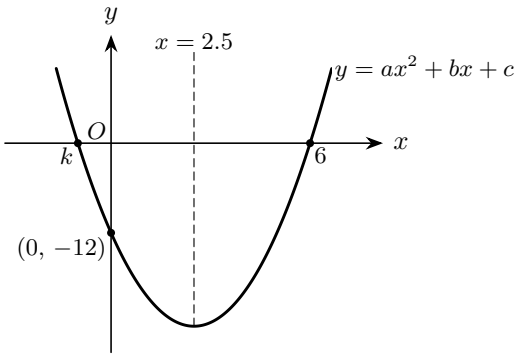
(a) $y = 7x - x^2$ (b) $y = x^2 - 6x - 7$ (c) $y = 25 - x^2$

Section 10: Finding the Equation of a Graph from a Sketch

32. The diagram shows the graph of $y = ax^2 + bx + c$. It cuts the x-axis at $(k, 0)$ and $(6, 0)$ and the y-axis at $(0, -12)$, and its line of symmetry is $x = 2.5$.

(a) Find the value of k.

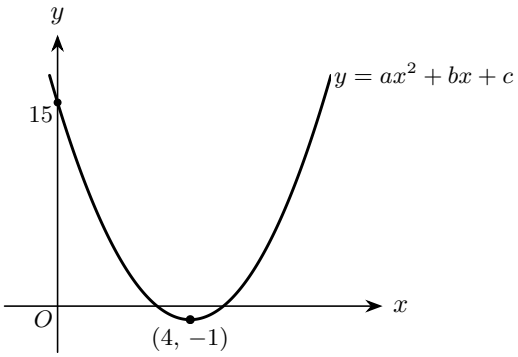
(b) Find the values of a, b and c.



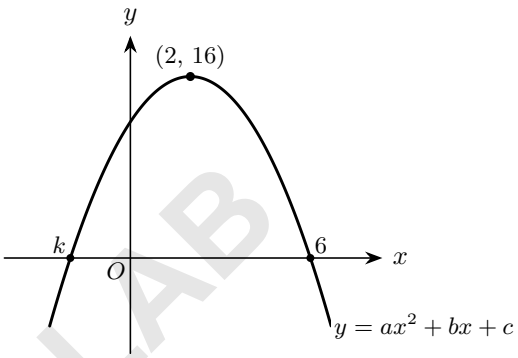
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33. The diagram shows the graph of $y = ax^2 + bx + c$. Its turning point is $(4, -1)$, and it cuts the y-axis at $(0, 15)$. Find the equation of the graph in the form $y = ax^2 + bx + c$.



34. The diagram shows the graph of $y = ax^2 + bx + c$. Its maximum point is $(2, 16)$, and it cuts the x-axis at $(k, 0)$ and $(6, 0)$. Find the values of k, a, b and c.



Supplementary Questions: Solving Equations Graphically

You will need graph paper for Questions 35–37. The method is shown at the end of the notes.

35. The variables x and y are connected by the equation $y = 2x^2 - 5x + 1$.

(a) Complete the table of values for $y = 2x^2 - 5x + 1$.

x	-1	0	1	2	3	4
y						

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2x^2 - 5x + 1$ for $-1 \leq x \leq 4$.

(c) Hence, solve the equation $2x^2 - 5x + 1 = 0$.

(d) By drawing a suitable straight line on your graph, solve the equation $x^2 - 3x + 1 = 0$.

36. The variables x and y are connected by the equation $y = x^2 + 6x + 9$.

(a) Complete the table of values for $y = x^2 + 6x + 9$.

x	-5	-4	-3	-2	-1	0
y						

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 6x + 9$ for $-5 \leq x \leq 0$.

(c) Hence, solve the equation $x^2 + 6x + 9 = 0$.

(d) By drawing a suitable straight line on your graph, solve the equation $x^2 + 5x + 3 = 0$.

37. (a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3 - 5x - x^2$ for $-6 \leq x \leq 1$.

(b) Use your graph to solve the equation $3 - 5x - x^2 = 0$.

(c) By drawing a suitable straight line on your graph, solve the equation $x^2 + 6x + 4 = 0$.

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Solutions to Exercises

Section 1 — Completing the square (Questions 1–3)

1(a) $x^2+10x+31=(x+5)^2-5^2+31=\boxed{(x+5)^2+6}$

1(b) $x^2-7x-2=\left(x-\frac{7}{2}\right)^2-\left(-\frac{7}{2}\right)^2-2=$
$\left(x-\frac{7}{2}\right)^2-\frac{49}{4}-\frac{8}{4}=\boxed{\left(x-\frac{7}{2}\right)^2-\frac{57}{4}}$

2(a) $x^2-1.4x+0.3=(x-0.7)^2-0.7^2+0.3=$
$(x-0.7)^2-0.49+0.3=\boxed{(x-0.7)^2-0.19}$

2(b) $x^2+\frac{2}{3}x+\frac{1}{9}=\left(x+\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2+\frac{1}{9}=$
$\left(x+\frac{1}{3}\right)^2-\frac{1}{9}+\frac{1}{9}=\boxed{\left(x+\frac{1}{3}\right)^2}$ (so $b=0$).

3(a)
$x^2-12x+11=(x-6)^2-36+11=\boxed{(x-6)^2-25}$
($a=6$, $b=25$).

3(b)
$x^2-\frac{7}{2}x+1=\left(x-\frac{7}{4}\right)^2-\frac{49}{16}+\frac{16}{16}=\boxed{\left(x-\frac{7}{4}\right)^2-\frac{33}{16}}$

3(c) $(x-7)(x+3)=x^2-4x-21=$
$(x-2)^2-4-21=\boxed{(x-2)^2-25}$ (so $a=2$, $b=-25$).

Section 2 — Coefficient of x^2 is -1 (Questions 4–5)

4 $-x^2+10x-7=-(x^2-10x)-7=$
$-\left[(x-5)^2-25\right]-7=\boxed{-(x-5)^2+18}$

5
$-x^2-3x+2=-(x^2+3x)+2=-\left[\left(x+\frac{3}{2}\right)^2-\frac{9}{4}\right]+2$
$=-\left(x+\frac{3}{2}\right)^2+\frac{9}{4}+\frac{8}{4}=\boxed{-\left(x+\frac{3}{2}\right)^2+\frac{17}{4}}$ ($h=-\frac{3}{2}$, $k=\frac{17}{4}$).

Section 3 — Factorisation (Questions 6–8)

6(a) $4x^2-10x=0$; $2x(2x-5)=0$ [take out common factor]; $\boxed{x=0 \text{ or } x=2\frac{1}{2}}$

6(b) $2x^2-7x-15=0$; $(2x+3)(x-5)=0$;
$\boxed{x=-1\frac{1}{2} \text{ or } x=5}$

7(a) $7^2-(3x)^2=0$ [a^2-b^2]; $(7+3x)(7-3x)=0$;
$\boxed{x=-2\frac{1}{3} \text{ or } x=2\frac{1}{3}}$

7(b) $(3x-1)^2-4^2=0$; $(3x-1-4)(3x-1+4)=0$;
$(3x-5)(3x+3)=0$; $\boxed{x=1\frac{2}{3} \text{ or } x=-1}$

8(a) $\times 4$: $4x^2+5x=6$; $4x^2+5x-6=0$;
$(4x-3)(x+2)=0$; $\boxed{x=\frac{3}{4} \text{ or } x=-2}$

8(b) Expand first: $x^2-x-6=14$; $x^2-x-20=0$;
$(x-5)(x+4)=0$; $\boxed{x=5 \text{ or } x=-4}$

8(c) $x(2x+1)-3(2x+1)=0$; $(2x+1)(x-3)=0$
[take out common factor $(2x+1)$; dividing by it would lose a solution]; $\boxed{x=-\frac{1}{2} \text{ or } x=3}$

Section 4 — Completing the square (Questions 9–11)

9(a)(i)
$x^2+9x+5=\left(x+\frac{9}{2}\right)^2-\frac{81}{4}+\frac{20}{4}=\boxed{\left(x+\frac{9}{2}\right)^2-\frac{61}{4}}$

9(a)(ii) $\left(x+\frac{9}{2}\right)^2=\frac{61}{4}$; $x+\frac{9}{2}=\pm\sqrt{\frac{61}{4}}$;
$x=-\frac{9}{2}\pm\sqrt{\frac{61}{4}}$
$\boxed{x=-0.595 \text{ or } x=-8.41}$

9(b) $(x-3)^2-9+4=0$; $(x-3)^2=5$; $x-3=\pm\sqrt{5}$;
$x=3\pm\sqrt{5}$
$\boxed{x=5.24 \text{ or } x=0.764}$

9(c) $\left(x-\frac{5}{4}\right)^2-\frac{25}{16}-\frac{64}{16}=0$; $\left(x-\frac{5}{4}\right)^2=\frac{89}{16}$;
$x=\frac{5}{4}\pm\sqrt{\frac{89}{16}}$
$\boxed{x=3.61 \text{ or } x=-1.11}$

10(a) $(x+2)^2-4-9=0$; $(x+2)^2=13$;
$x=-2\pm\sqrt{13}$; $\boxed{x=1.61 \text{ or } x=-5.61}$

10(b) $\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-\frac{28}{4}=0$; $\left(x-\frac{3}{2}\right)^2=\frac{37}{4}$;
$x=\frac{3}{2}\pm\sqrt{\frac{37}{4}}$
$\boxed{x=4.54 \text{ or } x=-1.54}$

11(i)
$x^2+\frac{2}{5}x-6=\left(x+\frac{1}{5}\right)^2-\frac{1}{25}-6=\boxed{\left(x+\frac{1}{5}\right)^2-\frac{151}{25}}$

11(ii) $\left(x+\frac{1}{5}\right)^2=\frac{151}{25}$; $x=-\frac{1}{5}\pm\sqrt{\frac{151}{25}}$;
$\boxed{x=2.26 \text{ or } x=-2.66}$

Section 5 — Quadratic formula (Questions 12–14)

12(a) $x=\dfrac{-(5)\pm\sqrt{(5)^2-4(1)(1)}}{2(1)}$;
$\boxed{x=-0.21 \text{ or } x=-4.79}$

12(b) $x=\dfrac{-(-11)\pm\sqrt{(-11)^2-4(5)(-6)}}{2(5)}$;
$\boxed{x=2.65 \text{ or } x=-0.45}$

13(a) $x^2-4x-11=0$;
$x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(-11)}}{2(1)}$;

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$\boxed{x = 5.87 \text{ or } x = -1.87}$

13(b) Cross multiply: $4x^2 = 3(2x+1)$;
$4x^2 - 6x - 3 = 0$;
$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(-3)}}{2(4)}$;
$\boxed{x = 1.90 \text{ or } x = -0.40}$

14(a) $x = \dfrac{-(4) \pm \sqrt{(4)^2 - 4(-3)(9)}}{2(-3)}$;
$\boxed{x = -1.19 \text{ or } x = 2.52}$

14(b) $x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(6)}}{2(3)}$;
$\boxed{x = 3.41 \text{ or } x = 0.586}$

14(c) $3x^2 - 5x - 28 = 0$;
$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-28)}}{2(3)} = \dfrac{5 \pm 19}{6}$;
$x = 4$ or $-2\frac{1}{3}$; $\boxed{x = 4 \text{ or } x = -2.33}$

Section 6 — Fractional equations (Questions 15–17)

15(a) $\times x$: $2x^2 + 15 = 11x$; $2x^2 - 11x + 15 = 0$;
$x = \dfrac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(15)}}{2(2)} = \dfrac{11 \pm 1}{4}$;
$\boxed{x = 3 \text{ or } x = 2\frac{1}{2}}$

15(b) $\dfrac{2(x+3)+3x}{x(x+3)} = \dfrac{2}{3}$; $\dfrac{5x+6}{x^2+3x} = \dfrac{2}{3}$;
$3(5x+6) = 2(x^2+3x)$;
$15x + 18 = 2x^2 + 6x$; $2x^2 - 9x - 18 = 0$;
$x = \dfrac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(-18)}}{2(2)} = \dfrac{9 \pm 15}{4}$;
$\boxed{x = 6 \text{ or } x = -1\frac{1}{2}}$

15(c) $\dfrac{(3x+1)-(x+2)}{(x+2)(3x+1)} = \dfrac{1}{10}$; $\dfrac{2x-1}{3x^2+7x+2} = \dfrac{1}{10}$;
$10(2x-1) = 3x^2 + 7x + 2$; $20x - 10 = 3x^2 + 7x + 2$;
$3x^2 - 13x + 12 = 0$;
$x = \dfrac{-(-13) \pm \sqrt{(-13)^2 - 4(3)(12)}}{2(3)} = \dfrac{13 \pm 5}{6}$;
$\boxed{x = 3 \text{ or } x = 1\frac{1}{3}}$

16(a) $\dfrac{2}{(x+3)(x-3)} + \dfrac{1}{x-3} = 3$ [factorise denominator];
$\dfrac{2+(x+3)}{(x+3)(x-3)} = 3$; $\dfrac{x+5}{x^2-9} = 3$; $x + 5 = 3x^2 - 27$;
$3x^2 - x - 32 = 0$; $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-32)}}{2(3)}$;
$\boxed{x = 3.44 \text{ or } x = -3.10}$

16(b) $\dfrac{7x+5}{(3x+1)(x-2)} - \dfrac{2}{x-2} = 1$ [factorise denominator];
$\dfrac{7x+5-2(3x+1)}{(3x+1)(x-2)} = 1$; $\dfrac{x+3}{3x^2-5x-2} = 1$;
$x + 3 = 3x^2 - 5x - 2$;
$3x^2 - 6x - 5 = 0$; $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-5)}}{2(3)}$;
$\boxed{x = 2.63 \text{ or } x = -0.633}$

16(c) $\dfrac{5}{3x-2} - \dfrac{x^2}{3x-2} = \dfrac{3}{2}$ (since $\dfrac{x^2}{2-3x} = -\dfrac{x^2}{3x-2}$);
$\dfrac{5-x^2}{3x-2} = \dfrac{3}{2}$; $2(5-x^2) = 3(3x-2)$; $10 - 2x^2 = 9x - 6$;
$2x^2 + 9x - 16 = 0$; $x = \dfrac{-(9) \pm \sqrt{(9)^2 - 4(2)(-16)}}{2(2)}$;
$\boxed{x = 1.36 \text{ or } x = -5.86}$

17(a) $\dfrac{(x+2)(x-1)-12}{4(x-1)} = 2$; $\dfrac{x^2+x-14}{4x-4} = 2$;
$x^2 + x - 14 = 8x - 8$;
$x^2 - 7x - 6 = 0$; $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-6)}}{2(1)}$;
$\boxed{x = 7.77 \text{ or } x = -0.772}$

17(b) $\dfrac{2m}{m+2} - \dfrac{3}{(m+2)(m-2)} = 1$;
$\dfrac{2m(m-2)-3}{(m+2)(m-2)} = 1$;
$2m^2 - 4m - 3 = m^2 - 4$; $m^2 - 4m + 1 = 0$;
$m = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$;
$\boxed{m = 3.73 \text{ or } m = 0.268}$

17(c) $\dfrac{3x^2-(x-1)(x+2)}{x(x+2)} = \dfrac{5}{x}$; $\dfrac{2x^2-x+2}{x(x+2)} = \dfrac{5}{x}$;
$\times x(x+2)$ $(x \neq 0)$: $2x^2 - x + 2 = 5(x+2)$;
$2x^2 - 6x - 8 = 0$; $x^2 - 3x - 4 = 0$;
$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)} = \dfrac{3 \pm 5}{2}$;
$\boxed{x = 4 \text{ or } x = -1}$

Section 7 — Word problems (Questions 18–25)

18(i) $\boxed{\frac{80}{x} \text{ h}}$ **(ii)** $\boxed{\frac{24}{x-45} \text{ h}}$

18(iii) $\dfrac{80}{x} + \dfrac{24}{x-45} = 2$; $\dfrac{80(x-45)+24x}{x(x-45)} = 2$;
$104x - 3600 = 2x^2 - 90x$;
$2x^2 - 194x + 3600 = 0$; $x^2 - 97x + 1800 = 0$ (Shown)

18(iv)
$x = \dfrac{-(-97) \pm \sqrt{(-97)^2 - 4(1)(1800)}}{2(1)} = \dfrac{97 \pm 47}{2}$;
$x = 72$ or 25.
$x = 25$ is rejected since the speed in town $x - 45$ must be positive.
Time in town $= \dfrac{24}{72-45} = \dfrac{8}{9}$ h $= \boxed{53.3 \text{ min}}$

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19(a) $\boxed{\frac{30}{x}\text{ h}}$ **(b)** $\boxed{\frac{30}{x-8}\text{ h}}$

19(c) $\frac{30}{x-8}-\frac{30}{x}=\frac{10}{60}$; $\frac{30x-30(x-8)}{x(x-8)}=\frac{1}{6}$;
$\frac{240}{x^2-8x}=\frac{1}{6}$;
$1440=x^2-8x$; $x^2-8x-1440=0$ (Shown)

19(d) $x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(-1440)}}{2(1)}$;
$\boxed{x=42.16 \text{ or } x=-34.16}$

19(e) $x=-34.16$ is rejected ($x>8$). Time back $=\frac{30}{42.158-8}=0.87828$ h $=52.70$ min $\approx \boxed{53\text{ min}}$

20 $\frac{240}{x-20}-\frac{240}{x}=\frac{3}{2}$; $\frac{240x-240(x-20)}{x(x-20)}=\frac{3}{2}$;
$\frac{4800}{x^2-20x}=\frac{3}{2}$;
$9600=3x^2-60x$; $x^2-20x-3200=0$ (Shown)
$x=\frac{-(-20)\pm\sqrt{(-20)^2-4(1)(-3200)}}{2(1)}$; $x=67.4$ or -47.4 (rejected, since a speed cannot be negative).
Average speed of the express bus $=\boxed{67.4\text{ km/h}}$

21 In 1 hour the tap fills $\frac{1}{x}$ of the tank and the outlet empties $\frac{1}{x+3}$ of it; together the tank gains $\frac{1}{12}$.
$\frac{1}{x}-\frac{1}{x+3}=\frac{1}{12}$; $\frac{(x+3)-x}{x(x+3)}=\frac{1}{12}$; $36=x^2+3x$;
$x^2+3x-36=0$ (Shown)
$x=\frac{-(3)\pm\sqrt{(3)^2-4(1)(-36)}}{2(1)}$; $x=4.6847$ or -7.68 (rejected, $x>0$).
Outlet alone: $x+3=\boxed{7.68\text{ h}}$

22(i) $\boxed{\frac{1}{x}}$ **(ii)** $\boxed{\frac{1}{x+4}}$

22(iii) $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{5}$; $\frac{2x+4}{x(x+4)}=\frac{1}{5}$;
$5(2x+4)=x^2+4x$;
$10x+20=x^2+4x$; $x^2-6x-20=0$ (Shown)

22(iv) $x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-20)}}{2(1)}$;
$\boxed{x=8.39 \text{ or } x=-2.39}$

22(v) $x=8.3852$ (-2.39 is rejected, since a time cannot be negative). In 2 hours the new machine prints $\frac{2}{8.3852}\times 100\%=\boxed{23.9\%}$

23(a) In 1 minute, Siti packs $\frac{48}{x}$ and Jun packs $\frac{30}{x+12}$ boxes; together they pack $\frac{102}{36}$.
$\frac{48}{x}+\frac{30}{x+12}=\frac{102}{36}$; $\frac{48(x+12)+30x}{x(x+12)}=\frac{17}{6}$;
$6(78x+576)=17(x^2+12x)$;
$468x+3456=17x^2+204x$;
$17x^2-264x-3456=0$ (Shown)

23(b) $x=\frac{-(-264)\pm\sqrt{(-264)^2-4(17)(-3456)}}{2(17)}=\frac{264\pm552}{34}$;
$\boxed{x=24 \text{ or } x=-8\tfrac{8}{17}}$

23(c) $x=24$ ($-8\frac{8}{17}$ is rejected, since a time cannot be negative). In 36 min: Siti packs $\frac{48}{24}\times 36=72$, Jun packs $\frac{30}{36}\times 36=30$.
Difference $=72-30=\boxed{42\text{ gift boxes}}$

24(a) $\boxed{(30-3t)\text{ cm}}$ **(b)** $\boxed{2t\text{ cm}}$

24(c) $\angle PBQ=90°$, so area of $\triangle PBQ=\frac{1}{2}\times PB\times BQ$:
$\frac{1}{2}(30-3t)(2t)=60$; $30t-3t^2=60$;
$3t^2-30t+60=0$; $\div 3$: $t^2-10t+20=0$ (Shown)

24(d) $t=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(20)}}{2(1)}$; $t=7.24$ or 2.7639.
Q reaches C when $2t=12$, i.e. $t=6$, so $t=7.24$ is rejected ($t\le 6$).
At $t=2.7639$: $PB=30-3(2.7639)=21.708$ cm, $BQ=2(2.7639)=5.5279$ cm.
$PQ=\sqrt{21.708^2+5.5279^2}=\boxed{22.4\text{ cm}}$

25(a) Planned share $=\$\frac{1350}{x}$; actual share $=\$\frac{1350}{x-3}$.
$\frac{1350}{x-3}-\frac{1350}{x}=75$; $\frac{1350x-1350(x-3)}{x(x-3)}=75$;
$4050=75x^2-225x$;
$\div 75$: $54=x^2-3x$; $x^2-3x-54=0$ (Shown)

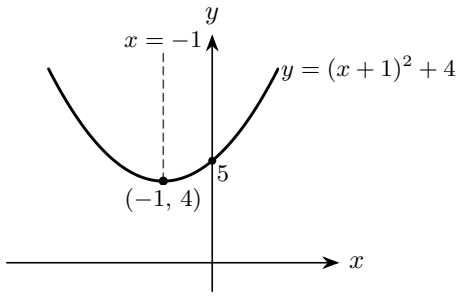
25(b) $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-54)}}{2(1)}=\frac{3\pm15}{2}$;
$\boxed{x=9 \text{ or } x=-6}$
$x=9$ (the number of friends cannot be negative). Each friend who went paid $\frac{1350}{9-3}=\boxed{\$225}$

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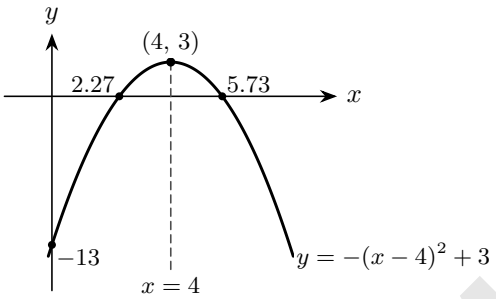
Section 8 — Sketching $y = \pm(x-h)^2 + k$ ***(Questions 26–28)***

26(a) (S) U-shaped; (T) min $(-1, 4)$; (Y) $y = 1 + 4$: $(0, 5)$; (X) $(x+1)^2 = -4$: no real solution, so no x-intercepts; (S) $x = -1$.



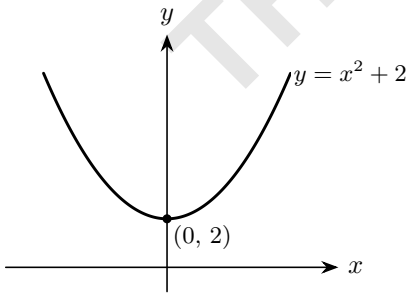
$\boxed{x = -1,\ \text{min } (-1, 4),\ y\text{-int } (0, 5),\ \text{no } x\text{-int}}$

26(b) (S) n-shaped; (T) max $(4, 3)$; (Y) $y = -16 + 3$: $(0, -13)$; (X) $(x-4)^2 = 3$, $x = 4 \pm \sqrt{3} = 5.73$ or 2.27; (S) $x = 4$.



$\boxed{x = 4,\ \text{max } (4, 3),\ (0, -13),\ (5.73, 0),\ (2.27, 0)}$

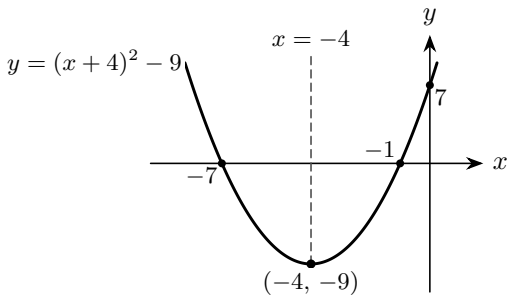
26(c) $y = (x-0)^2 + 2$: (S) U-shaped; (T) min $(0, 2)$, which is also the y-intercept; (X) $x^2 = -2$: no x-intercepts; (S) $x = 0$.



$\boxed{x = 0,\ \text{min } (0, 2),\ y\text{-int } (0, 2),\ \text{no } x\text{-int}}$

27(i) $x^2 + 8x + 7 = (x+4)^2 - 16 + 7 = \boxed{(x+4)^2 - 9}$ $(a = 4,\ b = -9)$.

27(ii) (S) U-shaped; (T) min $(-4, -9)$; (Y) $(0, 7)$; (X) $(x+4)^2 = 9$, $x + 4 = \pm 3$, $x = -1$ or -7; (S) $x = -4$.



28(a)

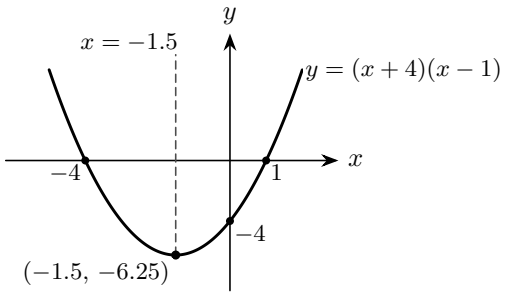
$x^2 - 12x + 40 = (x-6)^2 - 36 + 40 = \boxed{(x-6)^2 + 4}$

28(b) $\boxed{(6,\ 4)}$

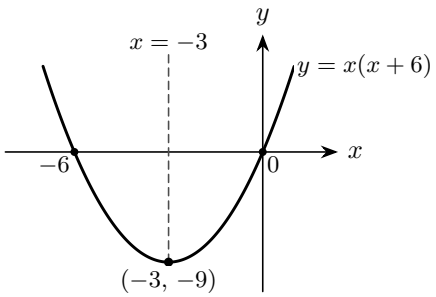
28(c) $(x-6)^2 \geq 0$ for all real x, so $x^2 - 12x + 40 = (x-6)^2 + 4 \geq 4 > 0$. It can never equal 0, so the equation has no real solutions: the minimum point $(6, 4)$ of the graph lies above the x-axis, so the graph never meets the x-axis.

Section 9 — Sketching $y = \pm(x-a)(x-b)$ ***(Questions 29–31)***

29(a) (S) U-shaped; (X) -4, 1; (S) $x = \frac{-4+1}{2} = -1.5$; (T) $y = (2.5)(-2.5) = -6.25$, min $(-1.5, -6.25)$; (Y) $y = (4)(-1)$: $(0, -4)$.



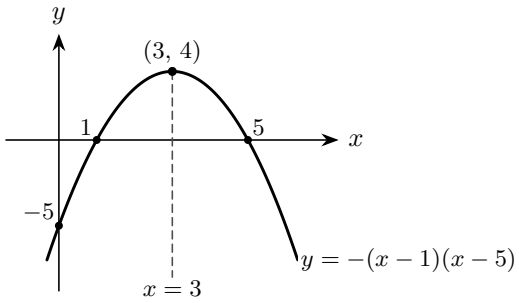
29(b) (S) U-shaped; (X) 0, -6; (S) $x = -3$; (T) $y = (-3)(3) = -9$, min $(-3, -9)$; (Y) $(0, 0)$.



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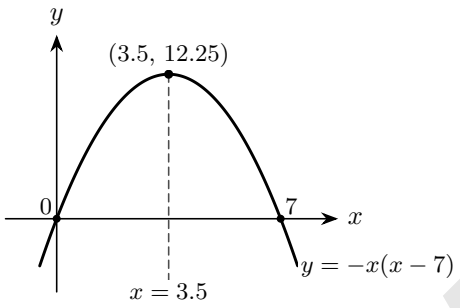
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29(c) (S) n-shaped; (X) 1, 5; (S) $x = 3$; (T) $y = -(2)(-2) = 4$, max $(3, 4)$; (Y) $y = -(-1)(-5)$: $(0, -5)$.

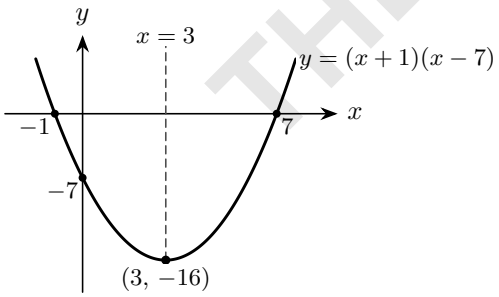


30 $y = -x^2 - x + 6 = -(x^2 + x - 6) = -(x + 3)(x - 2)$: the x-intercepts are -3 and 2.
$\boxed{a = -3,\ b = 2}$; y-intercept $\boxed{c = 6}$

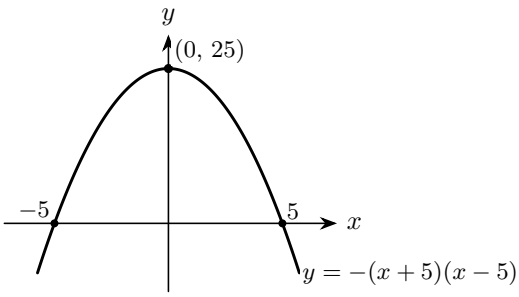
31(a) $y = 7x - x^2 = \boxed{-x(x - 7)}$. (S) n-shaped; (X) 0, 7; (S) $x = 3.5$; (T) $y = -3.5(3.5 - 7) = 12.25$, max $(3.5, 12.25)$; (Y) $(0, 0)$.



31(b) $y = x^2 - 6x - 7 = \boxed{(x + 1)(x - 7)}$. (S) U-shaped; (X) -1, 7; (S) $x = 3$; (T) $y = (4)(-4) = -16$, min $(3, -16)$; (Y) $(0, -7)$.



31(c) $y = 25 - x^2 = -(x^2 - 25) = \boxed{-(x + 5)(x - 5)}$. (S) n-shaped; (X) -5, 5; (S) $x = 0$; (T) max $(0, 25)$, also the y-intercept.



Section 10 — Equation from a sketch (Questions 32–34)

32(a) Line of symmetry: $\dfrac{k + 6}{2} = 2.5$; $k + 6 = 5$; $\boxed{k = -1}$

32(b) Equation: $y = a(x + 1)(x - 6)$. Sub $(0, -12)$: $-12 = a(0 + 1)(0 - 6)$; $a = 2$.
$y = 2(x + 1)(x - 6) = 2(x^2 - 5x - 6) = 2x^2 - 10x - 12$
$\boxed{a = 2,\ b = -10,\ c = -12}$

33 Since the turning point is $(4, -1)$, $y = a(x - 4)^2 - 1$. Sub $(0, 15)$: $15 = a(0 - 4)^2 - 1$; $16a = 16$; $a = 1$.
$y = (x - 4)^2 - 1 = x^2 - 8x + 16 - 1$; $\boxed{y = x^2 - 8x + 15}$

34 The line of symmetry passes through the maximum point, so it is $x = 2$: $\dfrac{k + 6}{2} = 2$; $\boxed{k = -2}$.
Equation: $y = a(x + 2)(x - 6)$. Sub $(2, 16)$:
$16 = a(2 + 2)(2 - 6)$; $-16a = 16$; $a = -1$.
$y = -(x + 2)(x - 6) = -(x^2 - 4x - 12) = -x^2 + 4x + 12$
$\boxed{a = -1,\ b = 4,\ c = 12}$

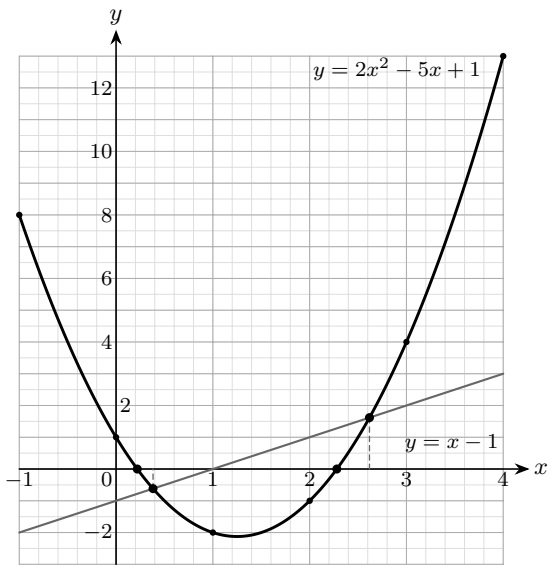
Supplementary Questions: Solving Equations Graphically (Questions 35–37)

Each curve is drawn through the plotted points of its table. To solve an equation with a straight line, rearrange it so that one side is the expression that was plotted; the other side gives the line to draw, and the solutions are the x-coordinates of the points where the line meets the curve. Readings are given to 1 decimal place; a reading within 0.1 of the value given is acceptable.

35(a)

x	-1	0	1	2	3	4
y	8	1	-2	-1	4	13

35(b) Plot the points and join them with a smooth curve.



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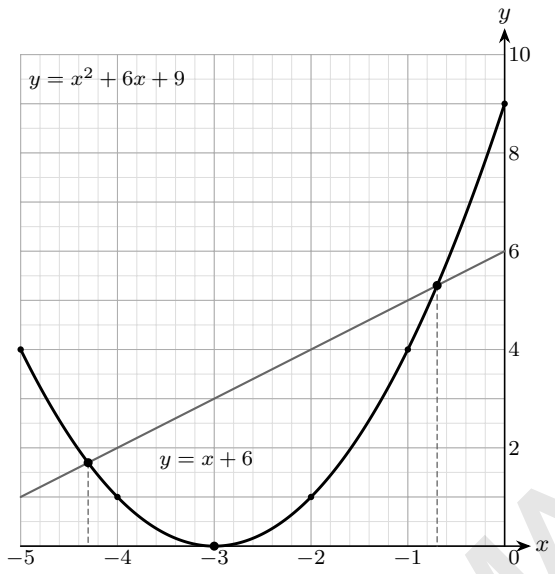
35(c) The curve meets the x-axis ($y = 0$) at
$\boxed{x = 0.2 \text{ or } x = 2.3}$

35(d) $x^2 - 3x + 1 = 0$; $\times 2$: $2x^2 - 6x + 2 = 0$;
$2x^2 - 5x + 1 = x - 1$ [add $x - 1$ to both sides].
Draw the line $y = x - 1$ through $(-1, -2)$ and $(4, 3)$.
It meets the curve at $\boxed{x = 0.4 \text{ or } x = 2.6}$

36(a)

x	-5	-4	-3	-2	-1	0
y	4	1	0	1	4	9

36(b) Plot the points and join them with a smooth curve.

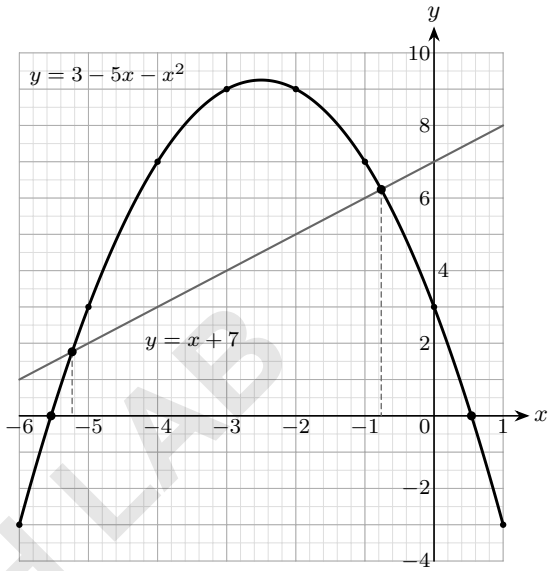


36(c) The curve touches the x-axis at one point only:
$\boxed{x = -3}$

36(d) $x^2 + 5x + 3 = 0$; $x^2 + 6x + 9 = x + 6$ [add $x + 6$ to both sides].
Draw the line $y = x + 6$ through $(-5, 1)$ and $(0, 6)$. It meets the curve at $\boxed{x = -4.3 \text{ or } x = -0.7}$

37(a) Table of values:

x	-6	-5	-4	-3	-2	-1	0	1
y	-3	3	7	9	9	7	3	-3



37(b) The curve meets the x-axis at
$\boxed{x = -5.5 \text{ or } x = 0.5}$

37(c) $x^2 + 6x + 4 = 0$; $-x^2 - 6x - 4 = 0$ [multiply by -1];
$3 - 5x - x^2 = x + 7$ [add $x + 7$ to both sides].
Draw the line $y = x + 7$ through $(-6, 1)$ and $(1, 8)$. It meets the curve at $\boxed{x = -5.2 \text{ or } x = -0.8}$

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