Inequalities

Linear Inequalities, Chained Inequalities, Largest and Smallest Values, Word Problems

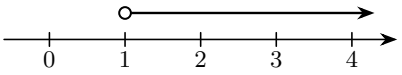
Secondary 3 — G3 Mathematics

Notes

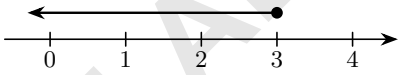
The Four Inequality Signs

$<$	is read 'is less than'	[$x < 7$: x is smaller than 7]
$\leq$	is read 'is less than or equal to'	[$x \leq 7$: x is smaller than 7, or equal to 7]
$>$	is read 'is greater than'	[$x > 7$: x is bigger than 7]
$\geq$	is read 'is greater than or equal to'	[$x \geq 7$: x is bigger than 7, or equal to 7]

On a number line, a **closed** (filled) circle shows that the end point belongs to the solution ($\leq$ or $\geq$), and an **open** (empty) circle shows that it does not ($<$ or $>$). The arrow means that the solution goes on without end in that direction.

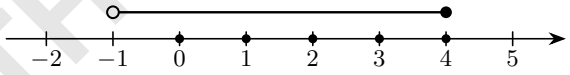


$x > 1$: open circle, 1 is not included



$x \leq 3$: closed circle, 3 is included

When x is an integer and $-1 < x \leq 4$, the possible values are $x = 0, 1, 2, 3, 4$.
The dots on the axis mark these integers. The open circle shows that -1 is left out, and the closed circle shows that 4 is kept.



Rules for Working with Inequalities

Adding or subtracting the same number on both sides **leaves the inequality sign unchanged**.

$$x + 4 \leq 9 \quad \Longrightarrow \quad x + 4 - 4 \leq 9 - 4 \quad \Longrightarrow \quad x \leq 5$$

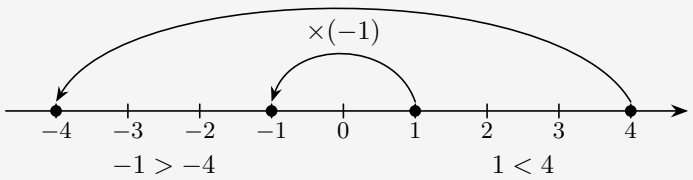
Multiplying or dividing both sides by the same **positive number** also **leaves the sign unchanged**.

$$\frac{x}{4} \geq 2 \quad \Longrightarrow \quad 4 \times \frac{x}{4} \geq 4 \times 2 \quad \Longrightarrow \quad x \geq 8$$

Multiplying or dividing both sides by the same **negative number reverses the sign**.

$$-3x \geq 12 \quad \Longrightarrow \quad \frac{-3x}{-3} \leq \frac{12}{-3} \quad \Longrightarrow \quad x \leq -4$$

Why must the sign be reversed? Multiplying by -1 flips every number across 0 on the number line, and this turns the order of any two numbers around: $1 < 4$, but $-1 > -4$.



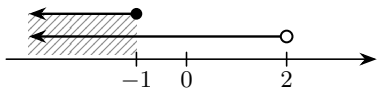
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Solving a chained inequality, or two inequalities together

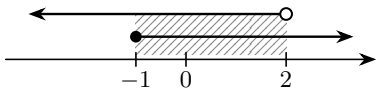
1. Solve **each** inequality on its own. A chained inequality is first broken into its two halves.
2. Show all of these solutions on **one number line**.
3. The solution is the part of the line that **every arrow covers**, where the arrows overlap. If the arrows have no part in common, there is no solution.

The overlap comes in three shapes. On each number line below, the hatched part is the solution.



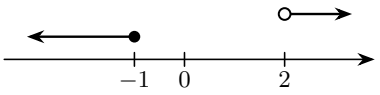
$x < 2$ and $x \leq -1$

same direction: $x \leq -1$



$x \geq -1$ and $x < 2$

facing each other: $-1 \leq x < 2$



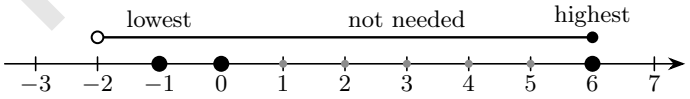
$x \leq -1$ and $x > 2$

facing away: no solution

Rules for finding a largest or smallest value

1. **From each given range, work only with its lowest and highest allowed values, together with 0 when 0 lies in the range.**
   Example: For $-2 < x \leq 6$, where x is an integer, the values to work with are -1, 0 and 6. The value -2 cannot be used, because the range leaves it out, and the values 1 to 5 are not needed, so leave them out of the working.
   Example: For $3 \leq x < 8$, where x is an integer, the values to work with are 3 and 7. The value 8 is left out by the range, and the values in between are not needed.
2. **Signs when multiplying or dividing: positive with positive gives a positive value**
   positive with negative gives a negative value
   negative with negative gives a positive value
3. **A square is never negative: it is positive, or 0 when the number squared is 0.**
   Example: $(-4)^2 = 16$ and $0^2 = 0$
4. **A cube has the same sign as the number that is cubed.**
   Example: $3^3 = 27$ and $(-3)^3 = -27$

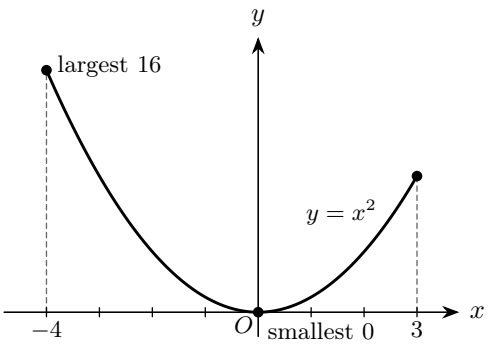
Rule 1 on a number line. For $-2 < x \leq 6$ with x an integer, the allowed integers run from -1 to 6. Only the two end values and 0 (the large dots) are used.



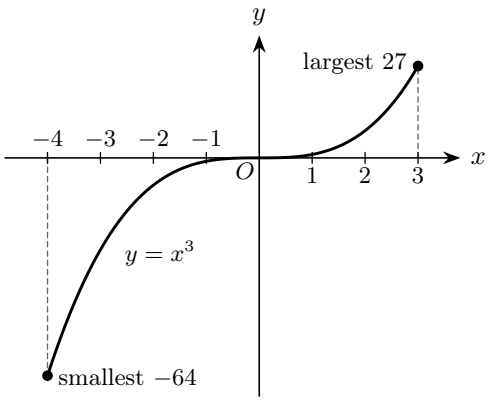
Rules 3 and 4 on a graph. Take $-4 \leq x \leq 3$. The smallest value of x^2 is 0, at $x = 0$, which is not an end of the range; the largest is at the end farther from 0, here $x = -4$. The cube x^3 has the same sign as x and always increases, so it is smallest at the lower limit and largest at the upper limit.

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$y = x^2$ for $-4 \le x \le 3$



$y = x^3$ for $-4 \le x \le 3$

A safe method when you are not sure. If you cannot see which values to use, list these values of each variable, keeping only those that lie inside its range:

- its largest and smallest **positive** values,
- its largest and smallest **negative** values,
- 0, if 0 lies in the range.

For example, if $-4 \le x < 9$, where x is an integer, try $x = -4,\ -1,\ 0,\ 1$ and 8 (the large dots). If x need not be an integer, use the two limits and 0, as in Rule 1; this works when both limits are included and no denominator can be 0.



Then use **trial and error**: work out the expression for each of these values (for every pair of values when there are two variables), and pick out the largest and the smallest of the results. It takes a little longer, but it is dependable when you are genuinely unsure.

Careful: you cannot divide by 0, so take care when you try $x = 0$: skip any value that makes a denominator 0.

Example. For $-4 \le x < 9$, where x is an integer, find the greatest and least values of $\dfrac{20}{x}$.

x	-4	-1	0	1	8
$\dfrac{20}{x}$	-5	-20	not allowed	20	2.5

Greatest value = 20 (at $x = 1$); least value = -20 (at $x = -1$). Using only the limit values -4 and 8 would have missed both.

One more check: when a bracket that contains both variables is squared, as in $(x - 3y)^2$, also ask whether the bracket can equal 0 for values inside the ranges. If it can, the smallest value is 0, even when none of the trial values gives 0.

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Key phrases. In a word problem, the wording decides which inequality sign to use.

is less than, is fewer than	$<$	is more than, exceeds	$>$
at most, not more than, cannot exceed	$\leq$	at least, not less than, a minimum of	$\geq$

between a and b inclusive $\quad a \leq \cdots \leq b$

Once the inequality is solved, answer the question that was actually asked. A number of people, coins or items has to be a whole number, so give the greatest or least whole number that satisfies the inequality.

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Exercises

Section 1: Recap of Sec 2 Inequalities

1. Solve each inequality and show its solution on a number line.

(a) $-5x \geq -35$

(b) $4x - \frac{3}{2}x \geq \frac{5}{6}$

(c) $\frac{7x}{4} - \frac{5}{6} < x$

(d) $0.4(5 - 3x) - 0.5(4 - 6x) < -9$

2. Solve each inequality and show its solution on a number line.

(a) $-\frac{1}{3}x \geq 5$

(b) $13x - 5x > -36$

(c) $\frac{1}{4}x + \frac{1}{6}x \geq -10$

3. Solve each inequality and show its solution on a number line.

(a) $\frac{1}{3}(2 - 6x) + x > 0$

(b) $\frac{9}{4}x > 2x + 5$

(c) $5(3 + 2x) - 3(5 - 4x) \leq -11$

(d) $\frac{1}{4}(5x - 12) + \frac{1}{3}(x + 9) \geq 19$

Section 2: Chained Inequalities

4. Solve each inequality, and show the answer on a number line.

(a) $2x + 5 < 33 - 4x < 3x + 12$

(b) $\frac{1}{3}x - 2 < 3(x + 2) < 4x + 1$

(c) $-5 < 8 - 3x \leq 20$, and write down the greatest and least integers that satisfy the inequality.

5. Solve each inequality, and show the answer on a number line.

(a) $-2 < \dfrac{x+1}{4} - \dfrac{x}{3} \leq 3$, and write down the greatest and least integers that satisfy the inequality.

(b) $-\frac{1}{4}\left(p + \frac{4}{3}\right) < \frac{1}{6}p \leq \frac{4}{3}$, and write down the prime numbers that satisfy the inequality.

(c) $\dfrac{2x}{5} - 4 \leq 3 < 3x - 7$, and write down the greatest rational number and the greatest prime number that satisfy the inequality.

(d) $-\dfrac{2x-5}{4} < \dfrac{5x-1}{8} \leq \dfrac{x+4}{3}$

6. Solve each inequality, and show the answer on a number line.

(a) $3 < 4x - 9 \leq 2x - 5$

(b) $2 - \dfrac{x+1}{4} \leq \dfrac{3x-5}{2} - x$, and state the smallest prime number that satisfies the inequality.

(c) $-3 \leq \dfrac{3(4x-2)}{5} < 18$, and write down the prime numbers that satisfy the inequality.

(d) $3n + 4 < 5n + 13 < \frac{1}{3}(2 - n)$

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Section 3: Largest and Smallest Values

7. Given that $2 \leq x \leq 5$ and $-4 \leq y \leq 2$, find the

(a) smallest possible value of $3y - 2x$,

(b) smallest possible value of $\dfrac{3y}{x} + xy$,

(c) greatest possible value of $x^2 - y^3$.

8. Given that x and y are integers such that $-3 \leq x < 4$ and $2 \leq y \leq 7$, find

(a) the least possible value of $y - x^2$,

(b) the greatest possible value of $\dfrac{y^2}{x+4}$.

9. Given that $1 \leq x \leq 4$ and $-6 \leq y \leq 5$, find

(a) the smallest value of xy,

(b) the smallest value of $\dfrac{y}{x}$,

(c) the smallest value of $(x - 2y)^2$.

Section 4: Word Problems

10. Tank A contains 900 litres of water and is drained at a steady 35 litres per minute. At the same moment, Tank B, which already contains 150 litres of water, starts to be filled at a steady 20 litres per minute.

(a) Let x be the number of minutes after both start. Write down the inequality in x that holds when Tank B contains more water than Tank A.

(b) Solve your inequality, and find the least whole number of minutes after which Tank B contains more water than Tank A.

11. A school sold 120 tickets for a concert. Of these, x were student tickets at \$6 each, and the rest were adult tickets at \$15 each. The total takings were more than \$1290. Form an inequality in x, and hence find the greatest possible number of student tickets sold.

12. A car park charges \$1.50 for the first hour and \$0.90 for each further 20 minutes or part of 20 minutes. Let x be the number of 20-minute periods (or parts of one) charged after the first hour. Form an inequality in x, and hence find the greatest number of minutes for which a car could have been parked if the charge was more than \$7 but less than \$8.

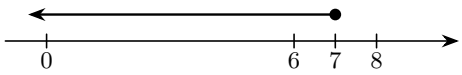
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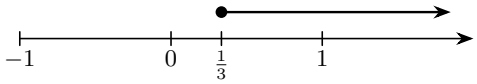
Solutions to Exercises

Section 1 — Recap of Sec 2 inequalities (Questions 1–3)

1(a) $-5x \geq -35$; $\boxed{x \leq 7}$ [divide by a negative number, so reverse the sign]

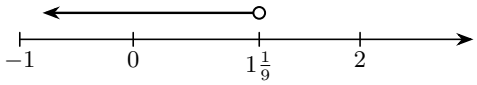


1(b) $\frac{5}{2}x \geq \frac{5}{6}$; $x \geq \frac{5}{6} \times \frac{2}{5}$; $\boxed{x \geq \frac{1}{3}}$

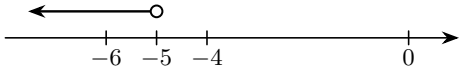


1(c) $\dfrac{3(7x)-2(5)}{12} < x$ [combine fractions using a common denominator]

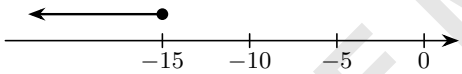
$21x - 10 < 12x$; $9x < 10$; $\boxed{x < 1\frac{1}{9}}$



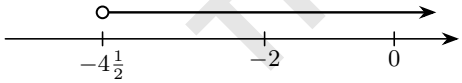
1(d) $2 - 1.2x - 2 + 3x < -9$; $1.8x < -9$; $\boxed{x < -5}$



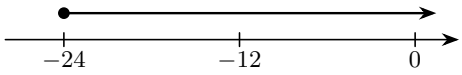
2(a) $-\frac{1}{3}x \geq 5$; $-x \geq 15$; $\boxed{x \leq -15}$ [multiply by a negative number, so reverse the sign]



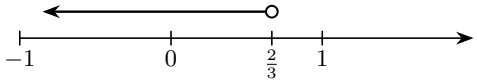
2(b) $8x > -36$; $\boxed{x > -4\frac{1}{2}}$



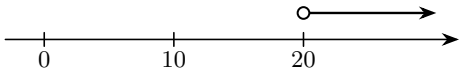
2(c) $\dfrac{3x+2x}{12} \geq -10$ [combine fractions]; $\dfrac{5x}{12} \geq -10$; $5x \geq -120$; $\boxed{x \geq -24}$



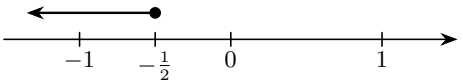
3(a) $\dfrac{2-6x+3x}{3} > 0$; $2 - 3x > 0$; $-3x > -2$; $\boxed{x < \frac{2}{3}}$



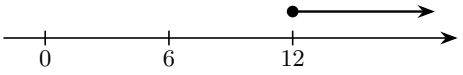
3(b) $9x > 8x + 20$ [multiply by 4]; $\boxed{x > 20}$



3(c) $15 + 10x - 15 + 12x \leq -11$; $22x \leq -11$; $\boxed{x \leq -\frac{1}{2}}$



3(d) $\dfrac{3(5x-12)+4(x+9)}{12} \geq 19$ [combine fractions]; $\dfrac{19x}{12} \geq 19$; $19x \geq 228$; $\boxed{x \geq 12}$



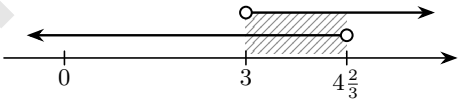
Section 2 — Chained inequalities (Questions 4–6)

4(a)

$2x + 5 < 33 - 4x$ and $33 - 4x < 3x + 12$

$6x < 28$ $\quad -7x < -21$

$x < 4\frac{2}{3}$ $\quad x > 3$



$\therefore$ $\boxed{3 < x < 4\frac{2}{3}}$

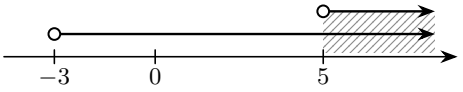
4(b)

$\frac{1}{3}x - 2 < 3(x+2)$ and $3(x+2) < 4x + 1$

$x - 6 < 9x + 18$ $\quad 3x + 6 < 4x + 1$

$-8x < 24$ $\quad -x < -5$

$x > -3$ $\quad x > 5$



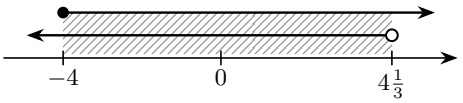
$\therefore$ $\boxed{x > 5}$

4(c)

$-5 < 8 - 3x$ and $8 - 3x \leq 20$

$3x < 13$ $\quad -3x \leq 12$

$x < 4\frac{1}{3}$ $\quad x \geq -4$



$\therefore$ $\boxed{-4 \leq x < 4\frac{1}{3}}$; greatest integer $\boxed{4}$, least integer $\boxed{-4}$

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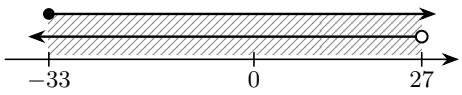
5(a) $\dfrac{x+1}{4} - \dfrac{x}{3} = \dfrac{3(x+1)-4x}{12} = \dfrac{3-x}{12}$ [combine fractions]

$-2 < \dfrac{3-x}{12}$ and $\dfrac{3-x}{12} \le 3$

$-24 < 3-x$ $\quad 3-x \le 36$

$-27 < -x$ $\quad -x \le 33$

$x < 27$ $\quad x \ge -33$



$\therefore$ $\boxed{-33 \le x < 27}$; greatest integer $\boxed{26}$, least integer $\boxed{-33}$

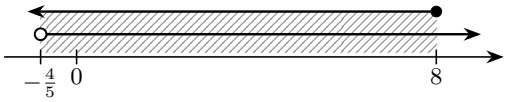
5(b) Multiply each part by 12:

$-3\left(p+\frac{4}{3}\right) < 2p$ and $2p \le 16$

$-3p - 4 < 2p$ $\quad p \le 8$

$-5p < 4$

$p > -\frac{4}{5}$



$\therefore$ $\boxed{-\frac{4}{5} < p \le 8}$; prime numbers: $\boxed{2,\ 3,\ 5,\ 7}$

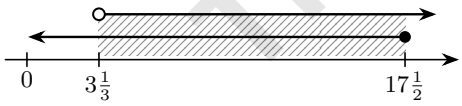
5(c)

$\dfrac{2x}{5} - 4 \le 3$ and $3 < 3x - 7$

$\dfrac{2x}{5} \le 7$ $\quad -3x < -10$

$2x \le 35$ $\quad x > 3\frac{1}{3}$

$x \le 17\frac{1}{2}$



$\therefore$ $\boxed{3\frac{1}{3} < x \le 17\frac{1}{2}}$; greatest rational number $\boxed{17\frac{1}{2}}$, greatest prime number $\boxed{17}$

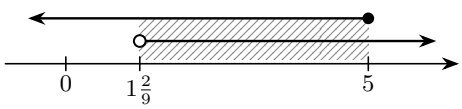
5(d) Multiply the left part by 8 and the right part by 24:

$-2(2x-5) < 5x-1$ and $3(5x-1) \le 8(x+4)$

$-4x + 10 < 5x - 1$ $\quad 15x - 3 \le 8x + 32$

$-9x < -11$ $\quad 7x \le 35$

$x > 1\frac{2}{9}$ $\quad x \le 5$



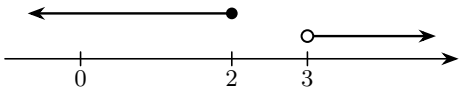
$\therefore$ $\boxed{1\frac{2}{9} < x \le 5}$

6(a)

$3 < 4x - 9$ and $4x - 9 \le 2x - 5$

$-4x < -12$ $\quad 2x \le 4$

$x > 3$ $\quad x \le 2$



The arrows do not overlap, so there is $\boxed{\text{no solution}}$.

6(b) $\dfrac{8-(x+1)}{4} \le \dfrac{3x-5-2x}{2}$ [combine fractions]

$\dfrac{7-x}{4} \le \dfrac{x-5}{2}$; $7 - x \le 2(x-5)$ [multiply by 4];

$7 - x \le 2x - 10$; $-3x \le -17$; $\boxed{x \ge 5\frac{2}{3}}$



Smallest prime number = $\boxed{7}$ [6 is not prime]

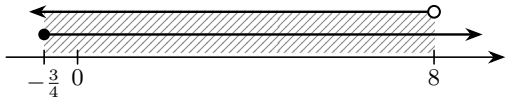
6(c)

$-3 \le \dfrac{3(4x-2)}{5}$ and $\dfrac{3(4x-2)}{5} < 18$

$-15 \le 12x - 6$ $\quad 12x - 6 < 90$

$-12x \le 9$ $\quad 12x < 96$

$x \ge -\dfrac{3}{4}$ $\quad x < 8$



$\therefore$ $\boxed{-\frac{3}{4} \le x < 8}$; prime numbers: $\boxed{2,\ 3,\ 5,\ 7}$

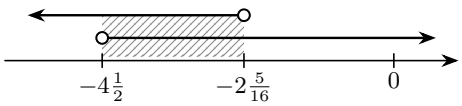
6(d)

$3n + 4 < 5n + 13$ and $5n + 13 < \frac{1}{3}(2-n)$

$-2n < 9$ $\quad 15n + 39 < 2 - n$

$n > -4\frac{1}{2}$ $\quad 16n < -37$

$n < -2\frac{5}{16}$



$\therefore$ $\boxed{-4\frac{1}{2} < n < -2\frac{5}{16}}$

Section 3 — Largest and smallest values (Questions 7–9)

7 Values of x: 2, 5; values of y: −4, 0, 2

(a) $3y - 2x = 3(-4) - 2(5) = -12 - 10 = \boxed{-22}$

(b) Use $y = -4$, so that both terms are negative, and try both values of x:

$x = 2$: $\dfrac{3(-4)}{2} + 2(-4) = -6 - 8 = -14$; $x = 5$:

$\dfrac{3(-4)}{5} + 5(-4) = -2\frac{2}{5} - 20$

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Smallest value = $\boxed{-22\frac{2}{5}}$

(c) $x^2 - y^3 = (5)^2 - (-4)^3 = 25 + 64 = \boxed{89}$ [largest x^2; $y = -4$ gives the most negative y^3, and subtracting it adds]

8 Values of x: $-3,\ 0,\ 3$; values of y: $2,\ 7$

(a) $y - x^2 = 2 - (-3)^2 = 2 - 9 = \boxed{-7}$ [smallest y, largest x^2; $x = 3$ gives the same x^2]

(b) $\dfrac{y^2}{x+4} = \dfrac{(7)^2}{-3+4} = \boxed{49}$ [largest numerator; $x = -3$ gives the smallest denominator, 1]

9 Values of x: $1,\ 4$; values of y: $-6,\ 0,\ 5$

(a) $xy = 4(-6) = \boxed{-24}$ [largest positive x, most negative y]

(b) $\dfrac{y}{x} = \dfrac{-6}{1} = \boxed{-6}$ [most negative y, smallest x]

(c) A square is never negative, so the smallest possible value is 0, if $x - 2y = 0$ can happen. It can: $x = 2y$ holds for $x = 2$, $y = 1$, and both lie in their ranges.

Smallest value = $(2 - 2(1))^2 = \boxed{0}$

Section 4 — Word problems (Questions 10–12)

10(a) $\boxed{150 + 20x > 900 - 35x}$

10(b) $55x > 750$; $x > 13\frac{7}{11}$. Least whole number of minutes = $\boxed{14}$

[check: after 14 minutes, Tank A contains $900 - 35(14) = 410$ litres and Tank B contains $150 + 20(14) = 430$ litres]

11 Number of adult tickets $= 120 - x$

$6x + 15(120 - x) > 1290$; $6x + 1800 - 15x > 1290$; $-9x > -510$; $x < 56\frac{2}{3}$

Greatest number of student tickets = $\boxed{56}$

12 $7 < 1.50 + 0.90x < 8$

$7 < 1.5 + 0.9x$	and	$1.5 + 0.9x < 8$
$0.9x > 5.5$		$0.9x < 6.5$
$x > 6\frac{1}{9}$		$x < 7\frac{2}{9}$

x is a whole number, so $x = 7$ (charge $1.50 + 7(0.90) = \$7.80$).

7 periods cover up to $7 \times 20 = 140$ minutes after the first hour.

Greatest time $= 60 + 140 = \boxed{200 \text{ minutes}}$

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