Indices, Standard Form and Interest

Supplementary Notes and Exercises

Secondary 3 — G3 Mathematics

Notes

Laws of Indices

Same base

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $a^m \div a^n = a^{m-n} \quad (a \neq 0)$

Same power

Law 3: $a^m \times b^m = (ab)^m$

Law 4: $a^m \div b^m = \left(\frac{a}{b}\right)^m \quad (b \neq 0)$

Power of a power

Law 5: $(a^m)^n = a^{m \times n} = a^{mn}$

Zero power

Law 6: $a^0 = 1 \quad (a \neq 0)$

Negative power

Law 7: $a^{-m} = \frac{1}{a^m} \quad (a \neq 0)$

Fractional power

Law 8: $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (a > 0)$

To solve equations

If $a^m = a^n$, then $m = n \quad (a > 0,\ a \neq 1)$.

If $a^m = b^m$, then $a = b \quad (a > 0,\ b > 0,\ m \neq 0)$.

Equations involving indices

Unknown in the index: write both sides as powers of the same base, then equate the indices: if $a^m = a^n$, then $m = n$.

Unknown in the base: make the power of x the subject, then undo the index by raising both sides to its reciprocal. When the index is a fraction, $x > 0$. When x is squared, there are two answers, one positive and one negative.

Sums and differences of powers: take out the lowest power as a common factor, so that the unknown appears in only one power; three equal terms add up to 3 times one of them.

Finding an unknown index: write each side as powers of the same bases, simplify with the laws of indices, then compare the indices of each base.

Sometimes, numbers can get too large, e.g. 326 890 000. Other times, numbers can get too small, e.g. 0.000 003 69. Therefore, writing numbers in the standard form can be useful:

$$A \times 10^n, \text{ where } n \text{ is an integer and } 1 \leq A < 10$$

When the decimal point shifts to the **left**, the power of 10 **increases** accordingly. When the decimal point shifts to the **right**, the power of 10 **decreases** accordingly.

Examples

$0.0\,0\,0\,2.5\,4 \times 10^{-1} = 2.54 \times 10^{-5}$

$35\,000 = 3.5 \times 10^4$

$0.0361 = 3.61 \times 10^{-2}$

$3406.88 \times 10^4 = 3.406\,88 \times 10^7$

Each arc moves the decimal point one place, from the open circle to the dot.

Common errors

0.267×10^5 is **not** standard form!

26.7×10^3 is **not** standard form!

2.67×10^4 is standard form!

- $\times$ and $\div$ in standard form: work on the numbers and on the powers of 10 separately, combining the powers of 10 with the laws of indices; then adjust so that $1 \leq A < 10$, e.g. $20 \times 10^{m+n} = 2 \times 10^{m+n+1}$.
- $+$ and $-$: write the numbers as ordinary numbers first, e.g. $(2.5 \times 10^3) - (5 \times 10^2) = 2500 - 500 = 2000 = 2 \times 10^3$. When a question says the calculator is not to be used, this line must be written.

To memorise: million $= 10^6$ billion $= 10^9$ trillion $= 10^{12}$

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Power of 10	Prefix	Symbol
10^{12}	tera / trillion	T
10^{9}	giga / billion	G
10^{6}	mega / million	M
10^{3}	kilo / thousand	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

Memorise the whole table.

Prefix conversion

Removing a prefix: replace the prefix by its power of 10.

$$20.5\,\text{gigabytes} = 20.5\,(10^{9})\,\text{bytes} = 2.05 \times 10^{10}\,\text{bytes}$$
$$1.5\,\text{nanoseconds} = 1.5\,(10^{-9})\,\text{seconds} = 1.5 \times 10^{-9}\,\text{seconds}$$

Inserting a prefix: divide by the power of 10 of the prefix.

$$5.2 \times 10^{5}\,\text{grams} = \frac{5.2 \times 10^{5}}{10^{3}}\,\text{kilograms} = 5.2 \times 10^{2}\,\text{kilograms}$$
$$3.2 \times 10^{-7}\,\text{metres} = \frac{3.2 \times 10^{-7}}{10^{-6}}\,\text{micrometres} = 3.2 \times 10^{-1}\,\text{micrometres}$$
$$3.2 \times 10^{-7}\,\text{metres} = \frac{3.2 \times 10^{-7}}{10^{-9}}\,\text{nanometres} = 3.2 \times 10^{2}\,\text{nanometres}$$

$$I = \frac{PRT}{100}$$

where I: the total simple interest incurred,
P: the principal amount,
R: the interest rate per annum,
T: the number of years.

Compound interest formula (given in the formula list of the examination paper):

$$A = P\left(1 + \frac{r}{100}\right)^{n}$$

where A: the total amount (interest + principal),
P: the principal amount,
r: the interest rate **per compounding period**,
n: the **number of compounding periods**.

If you find the different compounding periods difficult, use this formula instead.
Modified compound interest formula (**not** given in the examination):

$$A = P\left(1 + \frac{R}{100 \times k}\right)^{y \times k}$$

where A: the total amount (interest + principal),
P: the principal amount,
R: the interest rate per annum,
y: the number of years,
k: the **number of compounding periods in one year**.

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$k = 1$	compounded yearly	(the basic case)
$k = 2$	compounded half-yearly	(one year has 2 half-years)
$k = 3$	compounded every 4 months	(one year has 3 four-month periods)
$k = 4$	compounded quarterly	(one year has 4 quarters)
$k = 12$	compounded monthly	(one year has 12 months)

Things to check and highlight when reading the question (common errors)

1. Check whether the question asks for the **interest** or the **total amount**.
2. Check whether the question specifies the **number of s.f. or d.p.** to round off to. (If it is unspecified, the default for answers involving money is **2 d.p.**)

Simple interest vs compound interest

Simple interest: the interest earned every year is the same, because it is calculated as a percentage of the original principal.

Compound interest: the interest earned each year (or period) is counted in the principal for the following year (or period), to earn more interest.

Other compounding periods. If a year is divided into m equal compounding periods, then for a time of t years at $R\%$ per annum,

$$A = P\left(1 + \frac{R/m}{100}\right)^{mt}.$$

The whole time is compounded, even when t is not a whole number of years: for 4.5 years compounded annually, $n = 4.5$; compounded monthly, $n = 4.5 \times 12 = 54$. So divide the yearly rate, and multiply the number of years, by the number of compounding periods in a year:

Compounded	Periods in a year	r (per period)	n
annually	1	yearly rate	number of years
half-yearly	2	yearly rate $\div 2$	number of years $\times 2$
quarterly	4	yearly rate $\div 4$	number of years $\times 4$
monthly	12	yearly rate $\div 12$	number of years $\times 12$

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Exercises

Section 1: Laws of Indices

Simplifying with the laws of indices

1. Simplify each of the following.

(a) $(3x^5)^4 \times (-3x^5)^2$

(b) $(2x^2)^4 \times (4x^5)^2 \times (4x^2y)^0$

(c) $(5ab^6)(-8a^2b)^2 \div (4ab^2)^2$

(d) $(3ab^6)(2a^2b^{0.5})^2 \div (4ab^4)^2$

Zero, negative and fractional indices

2. Evaluate the following without using a calculator.

(a) $2^2 + 2^{-2} - 4^{\frac{1}{2}}$

(b) $2 \times \left(\dfrac{4^2 \times 8^3}{16^3}\right) + 3^0$

(c) $99^{50} \div 99^{49} \times \dfrac{1}{99}$

(d) $5^2 \times \left(8^0\right)^{-\frac{1}{2}} \times \left(\dfrac{25}{9}\right)^{-\frac{3}{2}}$

*No marks are awarded if the laws of indices are not used to obtain the answer.

3. Simplify each of the following, giving your answer in positive index form.

(a) $\dfrac{(xy^2)^{-2}}{(x^{-2}y^3)^{-3}}$

(b) $(-2x^5y^4)^3 \div (-4x^3y)^2$

(c) $(8x^2)^{\frac{1}{3}} \times (8x)^{-1} \div \left(8x^{\frac{1}{5}}\right)^0$

(d) $10x^3 \div (8x^{-6})^{\frac{2}{3}}$

(e) $\sqrt[3]{x^6y} \div \sqrt{y}$

(f) $\sqrt{4x^2y} \times 2x\sqrt{y^3}$

Solving equations in indices

4. Solve the following equations.

(a) $(243)^x = \dfrac{1}{81}$

(b) $7(7^x) = 1$

(c) $3^{2x} \times 27^{x-2} = 81$

Sums and differences of powers

5. Solve the equation $2^{x+3} = 320 - 2^{x+1}$.

Section 2: Introduction to Standard Form

6. Express the following numbers in standard form.

(a) 2650

(b) 89 000

(c) 0.768

(d) 100 000 000

(e) 0.000 0890

(f) 0.001 687

7. Express the following numbers in standard form.

(a) 5431

(b) 86 560

(c) 3 796 000

8. Express the following numbers in standard form.

(a) 89.1

(b) 0.001 23

(c) 0.000 0435

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Section 3: $+ \ - \ \times \ \div$ with Standard Form

9. Evaluate the following and give your answers in standard form.

(a) $9(3.2 \times 10^{-3})$ (b) $8(2.46 \times 10^{-4})$ (c) $3.5 \times 10^4 + 4.5 \times 10^5$

(d) $9.3 \times 10^{10} - 9.8 \times 10^8$ (e) $3.2 \times 10^4 \times 4 \times 10^9$ (f) $(8.64 \times 10^{10}) \div (3.6 \times 10^6)$

10. Given that $p = 3.6 \times 10^{-7}$ and $q = 22 \times 10^{-8}$, evaluate each of the following **without the use of a calculator**. Give each answer in standard form.

(a) $p + 2q$ (b) $p - q$ (c) $p \times q$

11. Given that the mass of substance A is 3.2×10^x grams and the volume of substance A is 4.25×10^y cm^3, find the density of substance A (in g/cm^3) and express your answer in terms of x and y and in standard form.

Section 4: Word Problems with Million, Billion and Other Prefixes

12. In 2021, the population of a city was 13.1 million. Express your answers to (a) and (b) in standard form.

(a) 900 000 of the population were at least 1.8 m tall. How many people were less than 1.8 m tall?

(b) 0.85% of the population were over 70 years old. How many people were over 70 years old?

(c) 5 years later, 5.1 million people left the city. What fraction of the total population remains in the city?

13. A hard disk drive can hold 1 terabyte of disk space.

(a) Express 1 terabyte in standard form.

(b) Given that a total of 456 gigabytes of disk space were used up, find, in bytes, the amount of disk space left. Express your answer in standard form.

14. The Earth is approximately 380 megametres from the Moon and approximately 150 gigametres from the Sun.

(a) At a particular instant, the Sun, the Moon and the Earth formed a straight line, with the Earth between the Sun and the Moon. Find the distance, in metres, between the Sun and the Moon, giving your answer in standard form, to a sensible degree of accuracy.

(b) Light travels 1.08 terametres in an hour. Estimate the time taken, in seconds, for light to travel from the Sun to the Earth, giving your answer to the nearest second.

15. The diameter of a spherical particle is 6 micrometres. Find, in metres and in standard form,

(a) the radius of the particle,

(b) the total volume of 1 million such particles, to 3 significant figures.

Section 5: Interest

Compound Interest

16. Paula invests \$2560 in a bank that pays compound interest at the rate of 4.5% per annum. If the interest is compounded monthly, find the total amount of money in her account at the end of the first year. Give your answer correct to the nearest cent.

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17. On 1 January 2030, Jack deposited \$500 into a new account. The account paid 8% per annum compound interest compounded annually. On 1 January 2031, he deposited another \$500 into the account.

(a) Find the total sum in the account immediately after this deposit.

(b) On 1 January 2032, he deposited a further \$500. Find the sum of money in the account immediately after this.

(c) On 1 January 2033, he deposited another \$500 into the account. He withdrew all the money in the account on 1 January 2034. Find, correct to the nearest cent, the sum of money he withdrew.

18. Mr Tan invested \$70 000 in an investment trust that pays R% compound interest per annum. After 4 years, he received \$18 373.39 interest. Find the value of R.

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Solutions to Exercises

Section 1 — Simplifying (Question 1)

1(a) $(3x^5)^4 \times (-3x^5)^2 = 81x^{20} \times 9x^{10} = \boxed{729x^{30}}$

1(b) $(2x^2)^4 \times (4x^5)^2 \times (4x^2y)^0 = 16x^8 \times 16x^{10} \times 1 = \boxed{256x^{18}}$

1(c) $(5ab^6)(-8a^2b)^2 \div (4ab^2)^2 = 5ab^6 \times 64a^4b^2 \div 16a^2b^4$
$= \dfrac{320a^5b^8}{16a^2b^4} = \boxed{20a^3b^4}$

1(d) $(3ab^6)(2a^2b^{0.5})^2 \div (4ab^4)^2 = 3ab^6 \times 4a^4b \div 16a^2b^8$
$= \dfrac{12a^5b^7}{16a^2b^8} = \boxed{\dfrac{3a^3}{4b}}$

Zero, negative and fractional indices (Questions 2–3)

2(a) $4 + \dfrac{1}{2^2} - \sqrt{4} = 4 + \dfrac{1}{4} - 2 = \boxed{2\tfrac{1}{4}}$

2(b) $2 \times \dfrac{(2^2)^2 \times (2^3)^3}{(2^4)^3} + 1 = 2 \times \dfrac{2^4 \times 2^9}{2^{12}} + 1$
$= 2 \times 2^1 + 1 = \boxed{5}$

2(c) $99^{50-49} \times 99^{-1} = 99^{1-1} = 99^0 = \boxed{1}$

2(d) $25 \times 1^{-\frac{1}{2}} \times \left(\dfrac{9}{25}\right)^{\frac{3}{2}} = 25 \times 1 \times \left(\dfrac{3}{5}\right)^3$
$= 25 \times \dfrac{27}{125} = \dfrac{27}{5} = \boxed{5\tfrac{2}{5}}$

3(a) $\dfrac{x^{-2}y^{-4}}{x^6y^{-9}} = x^{-8}y^5 = \boxed{\dfrac{y^5}{x^8}}$

3(b) $-8x^{15}y^{12} \div 16x^6y^2 = \boxed{-\dfrac{1}{2}x^9y^{10}}$

3(c) $8^{\frac{1}{3}}x^{\frac{2}{3}} \times \dfrac{1}{8x} \div 1 = \dfrac{2x^{\frac{2}{3}}}{8x} = \dfrac{1}{4}x^{-\frac{1}{3}} = \boxed{\dfrac{1}{4x^{\frac{1}{3}}}}$

3(d) $10x^3 \div 8^{\frac{2}{3}}x^{-4} = 10x^3 \div 4x^{-4} = \boxed{\dfrac{5x^7}{2}}$

3(e) $(x^6y)^{\frac{1}{3}} \div y^{\frac{1}{2}} = x^2y^{\frac{1}{3}-\frac{1}{2}} = x^2y^{-\frac{1}{6}} = \boxed{\dfrac{x^2}{y^{\frac{1}{6}}}}$

3(f) $(4x^2y)^{\frac{1}{2}} \times 2x(y^3)^{\frac{1}{2}} = 2xy^{\frac{1}{2}} \times 2xy^{\frac{3}{2}} = \boxed{4x^2y^2}$

Solving equations in indices (Question 4)

4(a) $\left(3^5\right)^x = \dfrac{1}{3^4} = 3^{-4}$; $5x = -4$; $\boxed{x = -\dfrac{4}{5}}$

4(b) $7^{1+x} = 7^0$; $1 + x = 0$; $\boxed{x = -1}$

4(c) $3^{2x} \times \left(3^3\right)^{x-2} = 3^4$
$3^{2x+3x-6} = 3^4$; $5x - 6 = 4$; $\boxed{x = 2}$

Sums and differences of powers (Question 5)

5 $2^{x+3} + 2^{x+1} = 320$
$2^x\left(2^3 + 2\right) = 320$; $10\,(2^x) = 320$
$2^x = 32 = 2^5$; $\boxed{x = 5}$

Sections 2–4 — Standard form (Questions 6–15)

The bracket gives the move of the decimal point.

6 (a) $2650 = \boxed{2.65 \times 10^3}$ [3 left]
(b) $89\,000 = \boxed{8.9 \times 10^4}$ [4 left]
(c) $0.768 = \boxed{7.68 \times 10^{-1}}$ [1 right]
(d) $100\,000\,000 = \boxed{1 \times 10^8}$ [8 left]
(e) $0.000\,0890 = \boxed{8.9 \times 10^{-5}}$ [5 right]
(f) $0.001\,687 = \boxed{1.687 \times 10^{-3}}$ [3 right]

7 (a) $5431 = \boxed{5.431 \times 10^3}$ [3 left]
(b) $86\,560 = \boxed{8.656 \times 10^4}$ [4 left]
(c) $3\,796\,000 = \boxed{3.796 \times 10^6}$ [6 left]

8 (a) $89.1 = \boxed{8.91 \times 10^1}$ [1 left]
(b) $0.001\,23 = \boxed{1.23 \times 10^{-3}}$ [3 right]
(c) $0.000\,0435 = \boxed{4.35 \times 10^{-5}}$ [5 right]

9(a) $9 \times 3.2 \times 10^{-3} = 28.8 \times 10^{-3} = \boxed{2.88 \times 10^{-2}}$

9(b) $8 \times 2.46 \times 10^{-4} = 19.68 \times 10^{-4} = \boxed{1.968 \times 10^{-3}}$

9(c) $35\,000 + 450\,000 = 485\,000 = \boxed{4.85 \times 10^5}$

9(d) $93\,000\,000\,000 - 980\,000\,000 = 92\,020\,000\,000$
$= \boxed{9.202 \times 10^{10}}$

9(e) $3.2 \times 4 \times 10^4 \times 10^9 = 12.8 \times 10^{13} = \boxed{1.28 \times 10^{14}}$

9(f) $\dfrac{8.64}{3.6} \times 10^{10-6} = \boxed{2.4 \times 10^4}$

10(a) $p + 2q = (3.6 \times 10^{-7}) + 2(22 \times 10^{-8})$
$= 0.000\,000\,36 + 0.000\,000\,44 = 0.000\,000\,80$
$= \boxed{8.0 \times 10^{-7}}$

10(b) $p - q = 0.000\,000\,36 - 0.000\,000\,22 = 0.000\,000\,14$
$= \boxed{1.4 \times 10^{-7}}$

10(c) $p \times q = 3.6 \times 22 \times 10^{-7} \times 10^{-8} = 79.2 \times 10^{-15}$
$= \boxed{7.92 \times 10^{-14}}$

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11 Density $= \dfrac{\text{mass}}{\text{volume}} = \dfrac{3.2 \times 10^{x}}{4.25 \times 10^{y}} = 0.752\,94 \times 10^{x-y}$

$= \boxed{7.53 \times 10^{x-y-1}\ \text{g/cm}^3}$ (to 3 s.f.)

12(a) $13\,100\,000 - 900\,000 = 12\,200\,000 = \boxed{1.22 \times 10^{7}}$

12(b) $\dfrac{0.85}{100} \times 13\,100\,000 = 111\,350 = \boxed{1.1135 \times 10^{5}}$

12(c) Remaining $= 13.1 - 5.1 = 8$ million

Fraction $= \dfrac{8}{13.1} = \boxed{\dfrac{80}{131}}$

13(a) 1 terabyte $= \boxed{1 \times 10^{12}\ \text{bytes}}$

13(b) 456 gigabytes $= 456 \times 10^{9}$ bytes

Left $= 1\,000\,000\,000\,000 - 456\,000\,000\,000$

$= 544\,000\,000\,000 = \boxed{5.44 \times 10^{11}\ \text{bytes}}$

14(a) $150 \times 10^{9} + 380 \times 10^{6} = 150\,380\,000\,000$ m

$= 1.5038 \times 10^{11} = \boxed{1.50 \times 10^{11}\ \text{m}}$ (to 3 s.f.)

14(b) Speed $= \dfrac{1.08 \times 10^{12}}{3600} = 3 \times 10^{8}$ m/s

Time $= \dfrac{150 \times 10^{9}}{3 \times 10^{8}} = \boxed{500\,\text{s}}$

15(a) Radius $= 3$ micrometres $= \boxed{3 \times 10^{-6}\ \text{m}}$

15(b) Volume $= 10^{6} \times \dfrac{4}{3}\pi\left(3 \times 10^{-6}\right)^{3} = 1.131 \times 10^{-10}$

$= \boxed{1.13 \times 10^{-10}\ \text{m}^3}$ (to 3 s.f.)

Section 5 — Compound interest (Questions 16–18)

16 $P = 2560$, $r = \dfrac{4.5}{12} = 0.375$, $n = 1 \times 12 = 12$

$A = 2560\left(1 + \dfrac{0.375}{100}\right)^{12} = 2677.6059\ldots = \boxed{\$2677.61}$

17 Year by year, after each deposit:

1 Jan 2031: $500 \times 1.08 + 500 = 1040$

1 Jan 2032: $1040 \times 1.08 + 500 = 1623.20$

1 Jan 2033: $1623.20 \times 1.08 + 500 = 2253.056$

1 Jan 2034: $2253.056 \times 1.08 = 2433.300\,48$

(a) $\boxed{\$1040}$ (b) $\boxed{\$1623.20}$ (c) $\boxed{\$2433.30}$

18 $A = 70\,000 + 18\,373.39 = 88\,373.39$

$\dfrac{88\,373.39}{70\,000} = \left(1 + \dfrac{R}{100}\right)^{4}$

$\sqrt[4]{\dfrac{88\,373.39}{70\,000}} = 1 + \dfrac{R}{100};\ 1.06 = 1 + \dfrac{R}{100}$

$\boxed{R = 6}$

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