Trigonometric Equations and Identities

Double Angle and Addition Formulae

Secondary 3 — G3 Additional Mathematics

Notes

Trigonometric Formulae (these are given in the examination)

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

Addition Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

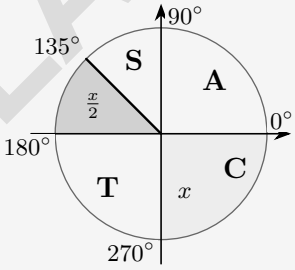
Double Angle Formulae

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Half Angle Formulae (these are not given in the examination, so learn them)

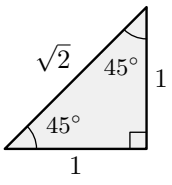
$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

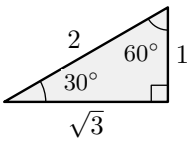
Note: each formula has a $\pm$ sign, but only one sign is correct. Find the quadrant that $\frac{x}{2}$ lies in, and keep the sign that the ratio has in that quadrant.



If $270° < x < 360°$, then $135° < \frac{x}{2} < 180°$. So $\frac{x}{2}$ is in the 2nd quadrant, where sin is positive and cos and tan are negative.

The two triangles give the exact values of sin, cos and tan of 30°, 45° and 60°. To find the exact value for another angle, write it as a sum or a difference of these angles and use an addition formula: for example, $75° = 45° + 30°$, $15° = 60° - 45°$ and $165° = 120° + 45°$.
For half of such an angle, such as $157.5° = \frac{1}{2} \times 315°$, use a half angle formula.
The formulae also work backwards. For example, $\cos 50° \cos 10° - \sin 50° \sin 10° = \cos(50° + 10°) = \cos 60° = \frac{1}{2}$, and $2 \sin 22.5° \cos 22.5° = \sin 45° = \frac{\sqrt{2}}{2}$.





For an angle in another quadrant, find its basic angle and use the sign that the ratio has in that quadrant. Leave exact answers in surd form, and rationalise any surd in a denominator.

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Where the half angle formulae come from

Start from a double angle formula for $\cos 2A$, replace A by $\frac{1}{2}A$ (so that $2A$ becomes A), and make the half angle the subject.

$$\cos 2A = 2\cos^2 A - 1 \qquad \cos 2A = 1 - 2\sin^2 A$$
$$\cos A = 2\cos^2\left(\tfrac{1}{2}A\right) - 1 \qquad \cos A = 1 - 2\sin^2\left(\tfrac{1}{2}A\right)$$
$$\frac{1+\cos A}{2} = \cos^2\left(\tfrac{1}{2}A\right) \qquad \frac{1-\cos A}{2} = \sin^2\left(\tfrac{1}{2}A\right)$$
$$\cos\left(\tfrac{1}{2}A\right) = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \sin\left(\tfrac{1}{2}A\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

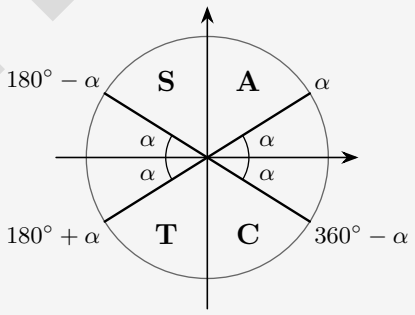
Dividing the second result by the first gives the formula for $\tan\left(\frac{1}{2}A\right)$.

You may be given one ratio of an angle and the quadrant the angle lies in. Draw a right-angled triangle in that quadrant and find the third side by Pythagoras' theorem. The horizontal and vertical sides take the signs of the axes they lie along (left and down are negative); the hypotenuse is always positive. You can then read off any other ratio of the angle. If two angles are said to be in the same quadrant, use the signs of the given ratios to decide which quadrant it is.

Take great care with the positive and negative signs. The double angle formulae work for any angle and its double: for example, $\sin 4A = 2\sin 2A\cos 2A$ and $\cos A = 1 - 2\sin^2\frac{A}{2}$.

To solve a trigonometric equation, use the double angle or addition formulae to turn it into **one ratio of one angle**. For $\cos 2x$, choose the form that matches the other terms: if the equation also has $\sin x$, use $\cos 2x = 1 - 2\sin^2 x$; if it has $\cos x$, use $\cos 2x = 2\cos^2 x - 1$. Then factorise, find the **basic angle** α from the positive value of the ratio, and use the signs in the four quadrants to list every angle in the range.

- Never cancel $\sin x$ or $\cos x$ from both sides. Bring every term to one side and factorise, or you will lose solutions.
- $\sin x$ and $\cos x$ always lie between -1 and 1, so reject a value such as $\sin x = \frac{4}{3}$.
- If every term is a $\sin^2 x$, $\sin x\cos x$ or $\cos^2 x$ term (write a constant k as $k(\sin^2 x + \cos^2 x)$), divide by $\cos^2 x$ to get a quadratic equation in $\tan x$.



In the 1st quadrant **A**ll three ratios are positive; in the 2nd only **S**in is positive, in the 3rd only **T**an, and in the 4th only **C**os.

When the angle is $ax + b$, for example $\tan(3x - 20°) = 2$:

1. Find the range of $ax + b$ from the range of x.
2. Find the basic angle α.
3. List every value of $ax + b$ given by the quadrants, marking any outside its range (N.A.).
4. Only now solve for x.

More tips for equations

- If the equation has $\tan 2x$ and $\tan x$, write $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$, multiply both sides by $1 - \tan^2 x$, bring every term to one side and factorise.
- If the equation has compound angles, such as $\sin(x + \beta)$ on one side and $\cos x$ or $\sin(x - \beta)$ on the other, expand each one with an addition formula. Collect the $\sin x$ terms and the $\cos x$ terms, then divide by $\cos x$ to get an equation in $\tan x$.
- In radians, the four angles in the diagram are α, $\pi - \alpha$, $\pi + \alpha$ and $2\pi - \alpha$.

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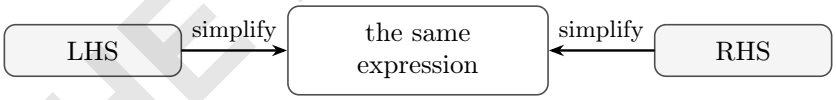
Proving identities: useful rules to try

1) **Look for a double angle formula.** Can you change $\sin 2x$ into $2\sin x\cos x$, or change $2\sin x\cos x$ back into $\sin 2x$?
2) **Look for an addition formula.** Is there a term such as $\cos(x-y)$, or an expression such as $\sin x\cos y+\cos x\sin y$?
3) **Start from the more complicated side.** If both sides are complicated, simplify each side until the two meet. Then write the working of one side in reverse under the working of the other.
4) Try changing $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ into $\sin x$ and $\cos x$.
5) **When two fractions are added or subtracted**, combine them over a common denominator.
6) **Look for a way to factorise and cancel.** Expand an expression only if its expansion contains $\sin^2 x+\cos^2 x$, $\sec^2 x-\tan^2 x$ or $\operatorname{cosec}^2 x-\cot^2 x$ (each equal to 1), or $-\sin^2 x-\cos^2 x$, $\tan^2 x-\sec^2 x$ or $\cot^2 x-\operatorname{cosec}^2 x$ (each equal to -1). For example, $(\sin x+\cos x)^2$ is worth expanding, because its expansion contains $\sin^2 x+\cos^2 x$:

$$(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x=1+2\sin x\cos x=1+\sin 2x.$$

7) **A useful simplification.** When a denominator is $1-\cos x$ or $1+\cos x$, multiply the numerator and the denominator by the other one of the two, so that the denominator becomes $\sin^2 x$. Do the same with $1-\sin x$ and $1+\sin x$, which give $\cos^2 x$:

$$\begin{aligned}
\frac{1}{1-\cos x}&=\frac{1+\cos x}{(1-\cos x)(1+\cos x)}=\frac{1+\cos x}{1-\cos^2 x}=\frac{1+\cos x}{\sin^2 x},\\
\frac{1}{1+\cos x}&=\frac{1-\cos x}{(1+\cos x)(1-\cos x)}=\frac{1-\cos x}{1-\cos^2 x}=\frac{1-\cos x}{\sin^2 x},\\
\frac{1}{1-\sin x}&=\frac{1+\sin x}{(1-\sin x)(1+\sin x)}=\frac{1+\sin x}{1-\sin^2 x}=\frac{1+\sin x}{\cos^2 x},\\
\frac{1}{1+\sin x}&=\frac{1-\sin x}{(1+\sin x)(1-\sin x)}=\frac{1-\sin x}{1-\sin^2 x}=\frac{1-\sin x}{\cos^2 x}.
\end{aligned}$$



Rule 3: when both sides are complicated, simplify each side until they meet. Then write the RHS working in reverse below the LHS working, so that the proof reads from LHS to RHS.

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Exercises

Give angles in degrees correct to 1 decimal place, and other answers correct to 3 significant figures, unless a question says otherwise.

Section 1: Special Angles

1. Find the exact value of each of the following. Leave your answers in surd form.

(a) $\cos 195°$ (b) $\cos 112.5°$

2. Without using a calculator, find the exact value of each of the following.

(a) $1 - 2\sin^2 67.5°$ (b) $\dfrac{\tan 70° - \tan 25°}{1 + \tan 70° \tan 25°}$

Section 2: Draw Triangle Questions

3. The angles A and B lie in the same quadrant, with $\sin A = \dfrac{20}{29}$ and $\tan B = -\dfrac{3}{4}$. Without using a calculator, evaluate

(a) $\cot(A + B)$ (b) $\operatorname{cosec}\left(\dfrac{B}{2}\right)$ (c) $\sin(180° + A)$

4. The angles x and y lie in the same quadrant, with $\cos x = -\dfrac{12}{13}$ and $\tan y = \dfrac{7}{24}$. Without using a calculator, find the exact value of

(a) $\cos(x + y)$ (b) $\tan(2x + 45°)$

5. The angle θ, measured in degrees, is obtuse and $\sin\theta = q$. Write each of the following as simply as possible in terms of q.

(a) $\tan 2\theta$ (b) $\tan\left(\dfrac{1}{2}\theta\right)$ (c) $\dfrac{\sin(90° - \theta)}{\tan\theta}$

Section 3: Solving Trigonometric Equations

6. Solve each of the following equations for $0° \leq x \leq 360°$.

(a) $2\sin^2 x + 3\cos x - 3 = 0$ (b) $6\sin x \cos x = 5\cos 2x$ (c) $\dfrac{4}{\sin^2 x} + \dfrac{3}{\tan x} = 5$

7. Solve each of the following equations for $-\pi \leq x \leq \pi$.

(a) $\sin 2x = \tan x$ (b) $2\sin\left(x + \dfrac{\pi}{4}\right) + 1 = 0$

Section 4: Proving Trigonometric Identities

8. Prove the following identities.

(a) $\dfrac{1 + \sin 2A - \cos 2A}{1 + \sin 2A + \cos 2A} \equiv \tan A$ (b) $\dfrac{\sin(A + B) - \sin(A - B)}{\cos(A + B) + \cos(A - B)} = \tan B$

9. Prove the following identities.

(a) $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A$ (b) $\dfrac{\sec x + \operatorname{cosec} x}{\sec x - \operatorname{cosec} x} = \dfrac{\tan x + 1}{\tan x - 1}$

10. (i) Prove the identity $\sin 3A \equiv 3\sin A - 4\sin^3 A$.

(ii) Hence, solve the equation $8\sin^3 A = 1 + 6\sin A$ for $270° \leq A \leq 360°$.

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11. (i) Show that $\dfrac{3\sin 2x - 4}{2 - 2\sin^2 x} = 3\tan x - 2\sec^2 x$.

(ii) Hence find all the angles between 0 and 2π which satisfy the equation $3\sin 2x - 4 = 14\sin^2 x - 14$.

12. Prove the following identities.

(a) $(\sin A - \cos A)(1 + \sin A \cos A) = \sin^3 A - \cos^3 A$

(b) $\dfrac{\cos A - \sin A}{\cos A + \sin A} + \dfrac{\cos A + \sin A}{\cos A - \sin A} = 2\sec 2A$

13. (i) Show that $\cos 3x - \cos x = -4\sin^2 x \cos x$.

(ii) Hence, solve the equation $\cos 3x + 2\cos x = 0$ for $0 \leq x \leq \pi$.

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Solutions to Exercises

Section 1 — Special Angles (Questions 1–2)

1(a) $\cos 195^\circ = \cos(150^\circ + 45^\circ)$

$$= \cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ$$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \boxed{-\frac{\sqrt{6}+\sqrt{2}}{4}}$$

1(b) $112.5^\circ = \frac{1}{2} \times 225^\circ$. Since $\cos 225^\circ = 2\cos^2(112.5^\circ) - 1$, $\cos 112.5^\circ = \pm\sqrt{\dfrac{1+\cos 225^\circ}{2}}$. Reject the positive sign, since 112.5° is in the 2nd quadrant.

$$\cos 112.5^\circ = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \boxed{-\frac{\sqrt{2-\sqrt{2}}}{2}}$$

2(a) Use $1 - 2\sin^2 A = \cos 2A$:

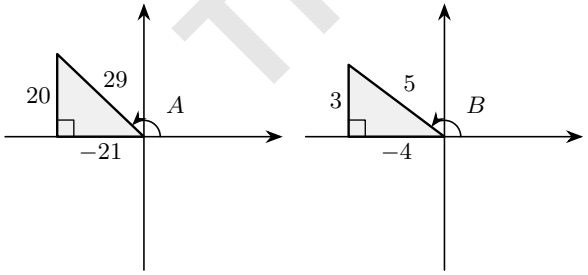
$$1 - 2\sin^2 67.5^\circ = \cos 135^\circ = -\cos 45^\circ = \boxed{-\frac{\sqrt{2}}{2}}$$

2(b) Use $\dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B)$:

$$\frac{\tan 70^\circ - \tan 25^\circ}{1 + \tan 70^\circ \tan 25^\circ} = \tan(70^\circ - 25^\circ) = \tan 45^\circ = \boxed{1}$$

Section 2 — Draw Triangle Questions (Questions 3–5)

3 $\sin A > 0$ (1st or 2nd quadrant) and $\tan B < 0$ (2nd or 4th quadrant), so both angles are in the 2nd quadrant.



$\cos A = -\frac{21}{29}$, $\tan A = -\frac{20}{21}$; $\sin B = \frac{3}{5}$, $\cos B = -\frac{4}{5}$

(a) $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{20}{21} - \frac{3}{4}}{1 - \left(-\frac{20}{21}\right)\left(-\frac{3}{4}\right)}$$

$$= \frac{-\frac{143}{84}}{\frac{24}{84}} = -\frac{143}{24}$$

$$\cot(A+B) = \boxed{-\frac{24}{143}}$$

(b) $\cos B = 1 - 2\sin^2\dfrac{B}{2}$, so

$$\sin\frac{B}{2} = \pm\sqrt{\frac{1-\cos B}{2}} = \pm\sqrt{\frac{1+\frac{4}{5}}{2}} = \pm\frac{3}{\sqrt{10}}.$$

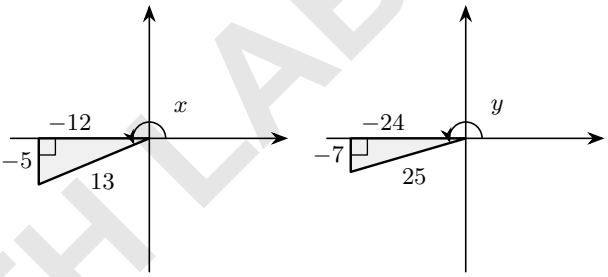
Reject negative, since $\frac{B}{2}$ is in the 1st quadrant ($90^\circ < B < 180^\circ$, so $45^\circ < \frac{B}{2} < 90^\circ$).

$\sin\dfrac{B}{2} = \dfrac{3}{\sqrt{10}}$, so $\operatorname{cosec}\dfrac{B}{2} = \boxed{\dfrac{\sqrt{10}}{3}}$

(c) $\sin(180^\circ + A) = \sin 180^\circ \cos A + \cos 180^\circ \sin A$

$$= 0 - \sin A = \boxed{-\frac{20}{29}}$$

4 $\cos x < 0$ (2nd or 3rd quadrant) and $\tan y > 0$ (1st or 3rd quadrant), so both angles are in the 3rd quadrant.



$\sin x = -\frac{5}{13}$, $\tan x = \frac{5}{12}$; $\sin y = -\frac{7}{25}$, $\cos y = -\frac{24}{25}$

(a) $$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= \left(-\frac{12}{13}\right)\left(-\frac{24}{25}\right) - \left(-\frac{5}{13}\right)\left(-\frac{7}{25}\right)$$

$$= \frac{288}{325} - \frac{35}{325} = \boxed{\frac{253}{325}}$$

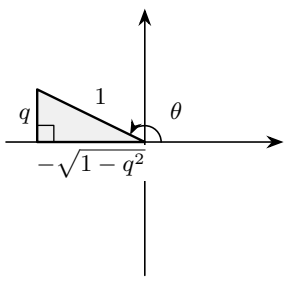
(b) $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{\frac{5}{6}}{1 - \frac{25}{144}}$$

$$= \frac{5}{6} \times \frac{144}{119} = \frac{120}{119}$$

$$\tan(2x + 45^\circ) = \frac{\tan 2x + \tan 45^\circ}{1 - \tan 2x \tan 45^\circ} = \frac{\frac{120}{119} + 1}{1 - \frac{120}{119}}$$

$$= \frac{\frac{239}{119}}{-\frac{1}{119}} = \boxed{-239}$$

5 θ is obtuse: draw the triangle in the 2nd quadrant. The adjacent side is $\sqrt{1-q^2}$, drawn to the left, so $\cos\theta = -\sqrt{1-q^2}$.



$$\tan\theta = -\frac{q}{\sqrt{1-q^2}}$$

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(a) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{-\frac{2q}{\sqrt{1-q^2}}}{1-\frac{q^2}{1-q^2}}$

$= -\dfrac{2q}{\sqrt{1-q^2}} \times \dfrac{1-q^2}{1-2q^2}$

$= \boxed{\dfrac{2q\sqrt{1-q^2}}{2q^2-1}}$

(b) $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$. Reject negative, since $45° < \frac{\theta}{2} < 90°$. Let $c = \sqrt{1-q^2}$, so $\cos\theta = -c$ and $1-c^2 = q^2$.

$\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1+c}{1-c}\times\dfrac{1+c}{1+c}} = \sqrt{\dfrac{(1+c)^2}{1-c^2}} = \dfrac{1+c}{q}$

$= \boxed{\dfrac{1+\sqrt{1-q^2}}{q}}$

(c) $\dfrac{\sin(90°-\theta)}{\tan\theta} = \dfrac{\cos\theta}{\frac{\sin\theta}{\cos\theta}} = \dfrac{\cos^2\theta}{\sin\theta} = \boxed{\dfrac{1-q^2}{q}}$

Section 3 — Solving Trigonometric Equations (Questions 6–7)

6(a) $2\big(1-\cos^2 x\big) + 3\cos x - 3 = 0$

$-2\cos^2 x + 3\cos x - 1 = 0$

$2\cos^2 x - 3\cos x + 1 = 0$

$(2\cos x - 1)(\cos x - 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = 1$

$\alpha = 60°$ or $\alpha = 0°$

$x = 60°,\ 360°-60°$ or $0°,\ 360°$

$\boxed{x = 0°,\ 60°,\ 300°,\ 360°}$

6(b) $3(2\sin x\cos x) = 5\cos 2x$, so $3\sin 2x = 5\cos 2x$,

$\tan 2x = \dfrac{5}{3}$

Range: $0° \le 2x \le 720°$

$\alpha = 59.04°$ (tan positive: Q1, Q3)

$2x = 59.04°,\ 239.04°,\ 419.04°,\ 599.04°$

$\boxed{x = 29.5°,\ 119.5°,\ 209.5°,\ 299.5°}$

6(c) $4\,\mathrm{cosec}^2 x + 3\cot x = 5$

$4\big(1+\cot^2 x\big) + 3\cot x - 5 = 0$

$4\cot^2 x + 3\cot x - 1 = 0$

$(4\cot x - 1)(\cot x + 1) = 0$

$\cot x = \frac{1}{4}$, $\tan x = 4$: $\alpha = 75.96°$,

$x = 75.96°,\ 180°+75.96°$

$\cot x = -1$, $\tan x = -1$: $\alpha = 45°$,

$x = 180°-45°,\ 360°-45°$

$\boxed{x = 76.0°,\ 135°,\ 256.0°,\ 315°}$

7(a) $2\sin x\cos x = \dfrac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x\big(2\cos^2 x - 1\big) = 0$

$\sin x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$

$\sin x = 0$: $x = -\pi,\ 0,\ \pi$

$\cos x = \pm\frac{1}{\sqrt{2}}$: $\alpha = \frac{\pi}{4}$, $x = \pm\frac{\pi}{4}$ (Q1, Q4) and $x = \pm\frac{3\pi}{4}$ (Q2, Q3)

$\boxed{x = -\pi,\ -\dfrac{3\pi}{4},\ -\dfrac{\pi}{4},\ 0,\ \dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \pi}$

7(b) $\sin\Big(x+\dfrac{\pi}{4}\Big) = -\dfrac{1}{2}$

Range: $-\frac{3\pi}{4} \le x + \frac{\pi}{4} \le \frac{5\pi}{4}$

$\alpha = \frac{\pi}{6}$ (sin negative: Q3, Q4)

$x + \frac{\pi}{4} = -\frac{\pi}{6},\ -\frac{5\pi}{6}$ (N.A.), $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$ (N.A.)

$x = -\frac{\pi}{6} - \frac{\pi}{4},\ \frac{7\pi}{6} - \frac{\pi}{4}$ $\quad \boxed{x = -\dfrac{5\pi}{12},\ \dfrac{11\pi}{12}}$

Section 4 — Proving Trigonometric Identities (Questions 8–13)

8(a) LHS $= \dfrac{(1-\cos 2A)+\sin 2A}{(1+\cos 2A)+\sin 2A}$

$= \dfrac{2\sin^2 A + 2\sin A\cos A}{2\cos^2 A + 2\sin A\cos A}$

$= \dfrac{2\sin A(\sin A+\cos A)}{2\cos A(\cos A+\sin A)}$

$= \tan A = \text{RHS}$

8(b) Numerator:

$(\sin A\cos B + \cos A\sin B) - (\sin A\cos B - \cos A\sin B)$

$= 2\cos A\sin B$

Denominator:

$(\cos A\cos B - \sin A\sin B) + (\cos A\cos B + \sin A\sin B)$

$= 2\cos A\cos B$

LHS $= \dfrac{2\cos A\sin B}{2\cos A\cos B} = \tan B = \text{RHS}$

9(a) RHS $= \dfrac{1}{\cos 2A} - \dfrac{\sin 2A}{\cos 2A} = \dfrac{1-\sin 2A}{\cos 2A}$

$= \dfrac{\cos^2 A + \sin^2 A - 2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$= \dfrac{(\cos A - \sin A)^2}{(\cos A-\sin A)(\cos A+\sin A)}$

$= \dfrac{\cos A - \sin A}{\cos A + \sin A} = \text{LHS}$

9(b) LHS $= \dfrac{\frac{1}{\cos x}+\frac{1}{\sin x}}{\frac{1}{\cos x}-\frac{1}{\sin x}} \times \dfrac{\sin x\cos x}{\sin x\cos x} = \dfrac{\sin x+\cos x}{\sin x-\cos x}$

$= \dfrac{\frac{\sin x}{\cos x}+1}{\frac{\sin x}{\cos x}-1} = \dfrac{\tan x+1}{\tan x-1} = \text{RHS}$

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10(i)

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$$
$$= 2\sin A \cos^2 A + \big(1 - 2\sin^2 A\big)\sin A$$
$$= 2\sin A\big(1 - \sin^2 A\big) + \sin A - 2\sin^3 A$$
$$= 3\sin A - 4\sin^3 A \text{ (proved)}$$

10(ii) $8\sin^3 A - 6\sin A = 1$, so

$-2\big(3\sin A - 4\sin^3 A\big) = 1$, $\sin 3A = -\dfrac{1}{2}$

Range: $810° \le 3A \le 1080°$

$\alpha = 30°$ (sin negative: Q3, Q4)

$3A = 570°$ (N.A.), $690°$ (N.A.), $930°$, $1050°$

$\boxed{A = 310°,\ 350°}$

11(i) $\text{LHS} = \dfrac{6\sin x \cos x - 4}{2\big(1 - \sin^2 x\big)} = \dfrac{6\sin x \cos x - 4}{2\cos^2 x}$

$$= \frac{3\sin x}{\cos x} - \frac{2}{\cos^2 x}$$
$$= 3\tan x - 2\sec^2 x = \text{RHS}$$

11(ii) $14\sin^2 x - 14 = -7\big(2 - 2\sin^2 x\big)$, so the equation is $\dfrac{3\sin 2x - 4}{2 - 2\sin^2 x} = -7$.

$$3\tan x - 2\sec^2 x = -7$$
$$3\tan x - 2\big(1 + \tan^2 x\big) = -7$$
$$2\tan^2 x - 3\tan x - 5 = 0$$
$$(2\tan x - 5)(\tan x + 1) = 0$$

$\tan x = \frac{5}{2}$: $\alpha = 1.1903$, $x = 1.1903,\ \pi + 1.1903$

$\tan x = -1$: $\alpha = \frac{\pi}{4}$, $x = \pi - \frac{\pi}{4},\ 2\pi - \frac{\pi}{4}$

$\boxed{x = 1.19,\ 2.36,\ 4.33,\ 5.50}$

12(a)

$$\text{LHS} = \sin A + \sin^2 A \cos A - \cos A - \sin A \cos^2 A$$
$$= \sin A\big(1 - \cos^2 A\big) - \cos A\big(1 - \sin^2 A\big)$$
$$= \sin^3 A - \cos^3 A = \text{RHS}$$

12(b)

$$\text{LHS} = \frac{(\cos A - \sin A)^2 + (\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$$
$$= \frac{2\cos^2 A + 2\sin^2 A}{\cos^2 A - \sin^2 A} = \frac{2}{\cos 2A}$$
$$= 2\sec 2A = \text{RHS}$$

13(i)

$$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$
$$= \big(1 - 2\sin^2 x\big)\cos x - 2\sin^2 x \cos x$$
$$= \cos x - 4\sin^2 x \cos x$$

So $\cos 3x - \cos x = -4\sin^2 x \cos x$ (shown).

13(ii) $\cos 3x + 2\cos x = (\cos 3x - \cos x) + 3\cos x$

$$= -4\sin^2 x \cos x + 3\cos x$$
$$= \cos x\big(3 - 4\sin^2 x\big) = 0$$

$\cos x = 0$: $x = \frac{\pi}{2}$

$\sin^2 x = \frac{3}{4}$, $\sin x = \pm\frac{\sqrt{3}}{2}$; for $0 \le x \le \pi$, $\sin x \ge 0$:

$x = \frac{\pi}{3},\ \pi - \frac{\pi}{3}$

$$\boxed{x = \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3}}$$

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