Quadratic Functions

Completing the Square, Graphs, Maximum and Minimum Values and Applications

Secondary 3 — G3 Additional Mathematics

Notes

The Completing-the-Square Formula

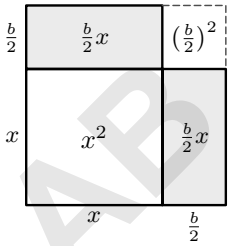
$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Examples

$x^2 + 10x$
$= (x+5)^2 - 5^2$
$= (x+5)^2 - 25$

$x^2 - 8x$
$= (x-4)^2 - (-4)^2$
$= (x-4)^2 - 16$

Halve the coefficient of x and put the half inside the bracket. Then subtract the square of that half.



Together the two shaded strips make bx. Adding the small corner square $\left(\frac{b}{2}\right)^2$ turns the shape into a square of side $x + \frac{b}{2}$.

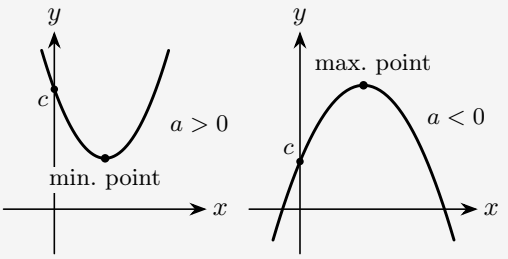
If the coefficient of x^2 is not 1, first take it out as a factor of the x^2 and x terms only. Complete the square inside the bracket, then multiply the bracket out again.

Recap

Every quadratic curve has an equation of the form $\boldsymbol{y = ax^2 + bx + c}$.
The number a, the coefficient of x^2, decides the shape.
The number c is where the curve cuts the y-axis.

If a is positive $(a > 0)$,
the curve is U-shaped and has a lowest point, the **minimum point**.

If a is negative $(a < 0)$,
the curve is n-shaped and has a highest point, the **maximum point**.

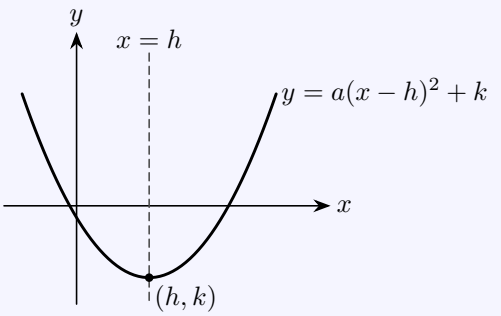


The Completed-Square Form

To sketch the curve, first complete the square, so that the equation reads $y = a(x-h)^2 + k$.

The turning point is (h, k)

- It is a minimum point if $a > 0$ and a maximum point if $a < 0$.
- The line of symmetry is the vertical line $x = h$.
- Put $x = 0$ to find where the curve cuts the y-axis.
- Put $y = 0$ to find where the curve cuts the x-axis (if it does).



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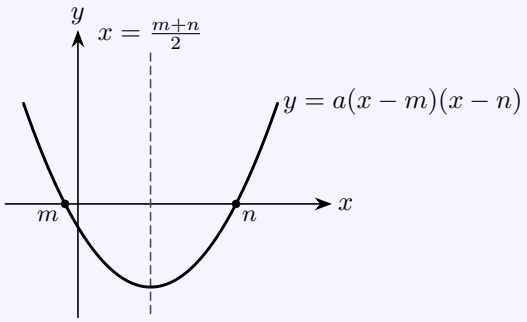
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If putting $y = 0$ gives $(x-h)^2$ equal to a negative number, the equation has no real solution, because a square can never be negative. The curve then does not meet the x-axis.

The Factorised Form

When the expression factorises, first factorise, so that the equation reads $y = a(x-m)(x-n)$.
Putting $y = 0$ gives $x = m$ or $x = n$, so:

the curve cuts the x-axis at $x = m$ and $x = n$



- The line of symmetry is halfway between them: $\boldsymbol{x = \dfrac{m+n}{2}}$.
- This is also the x-coordinate of the turning point. Substitute $x = \dfrac{m+n}{2}$ into the equation to get its y-coordinate.
- Put $x = 0$ to find the y-intercept.

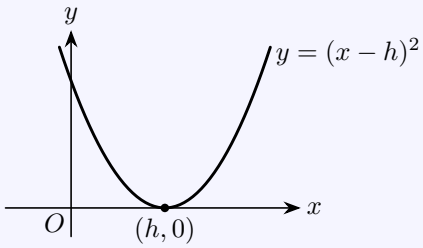
The numbers m and n are called the **roots** of the equation $a(x-m)(x-n) = 0$: they are its solutions.

To decide the shape of the curve, find the sign of the coefficient of x^2 when the brackets are expanded: positive gives a U-shaped curve, negative an n-shaped one. A bracket such as $(2x+1)$ or $(3-x)$ gives an x-intercept where the bracket equals 0.

Key Fact: A Square Is Never Negative

When a real number is squared, the answer is either zero or positive. It is never negative. So

$$x^2 \geq 0 \quad \text{for all values of } x$$
$$\text{and} \quad (x-h)^2 \geq 0 \quad \text{for all values of } x$$



To find the greatest or least value of a quadratic expression, complete the square first and then start from this fact.
The curve $y = (x-h)^2$ never goes below the x-axis. Its lowest point is $(h, 0)$, where it just touches the x-axis.

Justifying a greatest or least value. Complete the square to get $a(x-h)^2 + k$. Then build the expression up from $(x-h)^2 \geq 0$ one step at a time, giving the reason for each step: multiply both sides by a, then add k. Multiplying both sides by a negative number turns the inequality sign round.

- If $a > 0$, the expression is never less than k: its least value is k, when $x = h$.
- If $a < 0$, the expression is never greater than k: its greatest value is k, when $x = h$.

The same steps show that an expression is always positive, always negative, or never on the wrong side of a given number, for all values of x.

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To find the equation of a quadratic curve from its sketch, start from what the sketch tells you.

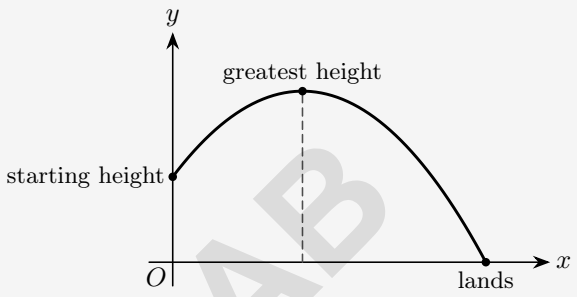
- If the curve cuts the x-axis at $x = m$ and $x = n$, start with $y = a(x - m)(x - n)$.
- If the turning point is (h, k), start with $y = a(x - h)^2 + k$.

Then put in the coordinates of one more point on the curve (often the point where it meets the y-axis) to find a. Expand the brackets if the answer is needed in the form $y = ax^2 + bx + c$.

Quadratic functions are also used to model real situations: the flight of an object through the air, a hanging cable or an arch, and the cost or profit of a business.

Parabolic motion. When the height of a moving object is a quadratic function of the horizontal distance x (or of the time t), its graph is an n-shaped curve. Each question about the flight is a question about one feature of this curve:

- **starting height**: put $x = 0$ (the y-intercept);
- **greatest height**: complete the square (the maximum point);
- **landing point**: put $y = 0$ and solve, keeping the positive root (the x-intercept).



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Exercises

Unless a question says otherwise, give non-exact answers correct to 3 significant figures.

Section 1: Completing the Square

1. Write each expression in the form $(x+a)^2+b$.

(a) $x^2-14x-5$ (b) $x^2-9x+12$

2. By completing the square, write each expression in the form $a(x+b)^2+c$.

(a) $5x^2-10x+3$ (b) $1-2x^2-6x$

3. Complete the square for each expression, giving your answer in the form $a(x+b)^2+c$.

(a) $2x^2+10x+9$ (b) $\frac{1}{3}x^2+2x-1$

Section 2: Sketching $y=a(x-h)^2+k$

4. Write each equation in the form $y=a(x-p)^2+q$. Hence state the coordinates of the turning point and say whether it is a maximum point or a minimum point.

(a) $y=-2x^2+7x+4$ (b) $y=4x^2-24x+31$

5. Write $f(x)=-x^2+10x-4$ in the form $f(x)=a(x+p)^2+q$. Hence state the greatest value of $f(x)$ and the value of x at which it occurs.

6. Sketch the graph of $y=-5+6x-3x^2$, marking clearly the y-intercept, the coordinates of the maximum point and the line of symmetry.

Section 3: Sketching $y=a(x-m)(x-n)$

7. Sketch each graph. Mark the points where it meets the x- and y-axes and the line of symmetry, and state the coordinates of the turning point.

(a) $y=2(2x+3)(x-1)$ (b) $y=(2x+1)(5-x)$

8. For each equation, sketch the curve, showing the x- and y-intercepts and the line of symmetry, and give the coordinates of the maximum or minimum point.

(a) $y=5x-2x^2$ (b) $y=16-x^2$

Section 4: Maximum and Minimum Values

9. Show that $27+24x+4x^2\geq -9$ for all values of x.

10. Explain why $-x^2-8x-21$ is negative for all values of x.

11. Find the greatest value of $-3x^2+2x+5$. Give a reason for your answer.

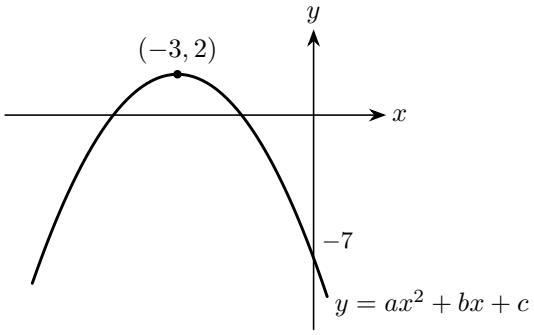
12. Find the least value of $x^2+8x+20$. Hence find the range of possible values of $\dfrac{1}{x^2+8x+20}$.

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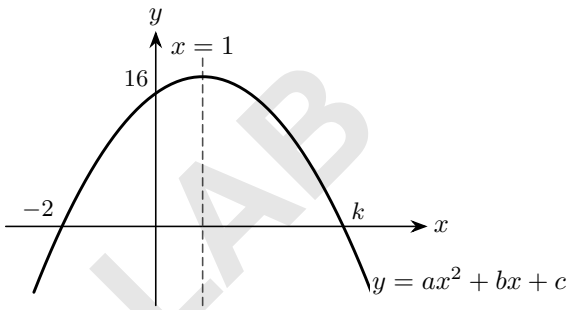
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Section 5: Finding the Equation of a Quadratic Curve

13. The diagram shows the graph of $y = ax^2 + bx + c$. The turning point is $(-3, 2)$ and the curve meets the y-axis at $(0, -7)$. Find the equation of the graph in the form $y = ax^2 + bx + c$.



14. The diagram shows the graph of $y = ax^2 + bx + c$. The curve meets the y-axis at $(0, 16)$ and the x-axis at $(-2, 0)$ and $(k, 0)$. The line of symmetry is $x = 1$. Find the values of k, a, b and c.



15. (i) A quadratic curve has its turning point at $(3, 5)$ and passes through the point $(1, 13)$. Find the equation of the curve.

(ii) A second quadratic curve has equation $y = -\frac{1}{2}x^2 - x + 4$. Write this equation in the form $y = a(x - h)^2 + k$, where a, h and k are constants.

(iii) Use your answers to (i) and (ii) to explain why the two curves do not meet.

Section 6: Application Questions

16. A volleyball player serves the ball. The height, H metres, of the ball above the ground can be modelled by $H = 2.2 + 0.6x - 0.05x^2$, $x \geq 0$, where x m is the horizontal distance of the ball from the server. The ball moves in a vertical plane.

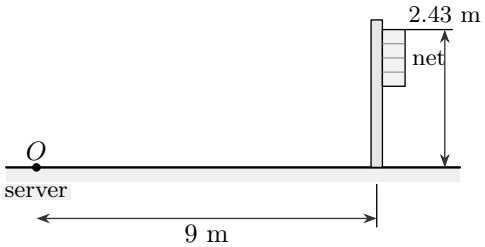
(i) With reference to the model, explain what the constant 2.2 represents.

(ii) Express $2.2 + 0.6x - 0.05x^2$ in the form $a(x - h)^2 + k$, where a, h and k are constants to be found.

(iii) What is the greatest height reached by the ball?

The net is 9 m from the server, measured horizontally, and the top of the net is 2.43 m above the ground, as shown in the diagram.

(iv) Will the ball pass over the net? Justify your answer.



17. A firework is launched straight upwards from a platform. Its height above the ground, y m, t seconds after it is launched is given by $y = -5t^2 + 30t + c$, where c m is the height of the platform.

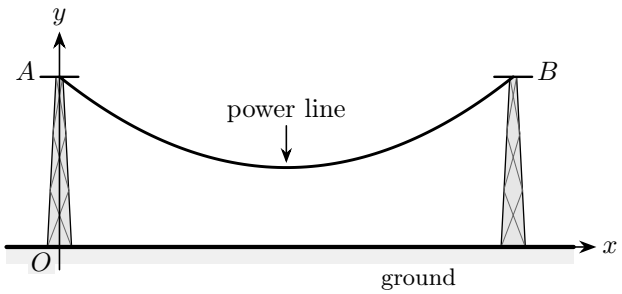
(i) By completing the square, find the range of values of c for which the firework does not reach a height of 60 m.

(ii) Given that $c = 10$, express y in the form $y = a(t - h)^2 + k$. Hence find the coordinates of the maximum point.

(iii) Explain how your answer to part (i) agrees with your answer to part (ii).

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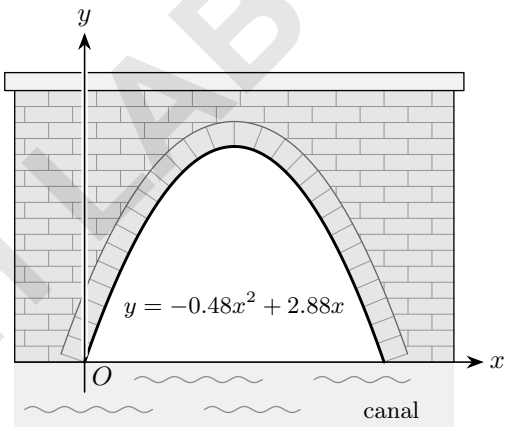
18. An electric power line hangs between two pylons, A and B, which are 30 m tall and 120 m apart. The line sags in the shape of a parabola, and its lowest point is 14 m above the ground. The height of the line above the ground is y m and the horizontal distance from pylon A is x m.



(i) Find a quadratic function in the form $y = a(x - h)^2 + k$ that models the power line.

(ii) Find the horizontal distances from pylon A at which the power line is 18 m above the ground.

19. The diagram shows an arched bridge over a canal. The underside of the arch can be modelled by $y = -0.48x^2 + 2.88x$, where y m is the height above the water and x m is the horizontal distance from O, at one side of the arch.



(i) Express y in the form $a(x - b)^2 + c$, where a, b and c are constants.

(ii) Hence or otherwise, state the coordinates of the highest point of the arch.

(iii) A boat is 2.4 m wide, and its top is 3.7 m above the water. Can the boat pass under the bridge? Justify your answer with calculations.

20. A choir holds a charity concert. When each ticket costs \$$x$, the money raised, \$$y$, after paying for the hire of the hall, is modelled by $y = -20x^2 + 1000x - 6000$. Ticket prices are whole numbers of dollars.

(i) Explain the meaning of "-6000" in the model.

(ii) What is the lowest ticket price at which the concert raises money?

(iii) What ticket price raises the most money? How much money is raised at this price?

21. A farmer uses 60 m of fencing to make a rectangular pen. The width of the pen is x m.

(i) Find an expression, in terms of x, for the area, A m^2, of the pen.

(ii) Find the x-intercepts of the graph of A against x.

(iii) Find the greatest area that the pen can have.

(iv) Show that the greatest area happens only when the pen is a square.

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Solutions to Exercises

Section 1 — Completing the square (Questions 1–3)

1(a) $x^2 - 14x - 5 = (x-7)^2 - 7^2 - 5$

$= \boxed{(x-7)^2 - 54}$

1(b) $x^2 - 9x + 12 = \left(x - \frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 12$

$= \boxed{\left(x - \frac{9}{2}\right)^2 - \frac{33}{4}}$

2(a) $5x^2 - 10x + 3 = 5\left[x^2 - 2x\right] + 3$

$= 5\left[(x-1)^2 - 1^2\right] + 3$

$= 5(x-1)^2 - 5 + 3$

$= \boxed{5(x-1)^2 - 2}$

2(b) $1 - 2x^2 - 6x = -2\left[x^2 + 3x\right] + 1$

$= -2\left[\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] + 1$

$= -2\left(x + \frac{3}{2}\right)^2 + \frac{9}{2} + 1$

$= \boxed{-2\left(x + \frac{3}{2}\right)^2 + \frac{11}{2}}$

3(a) $2x^2 + 10x + 9 = 2\left[x^2 + 5x\right] + 9$

$= 2\left[\left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right] + 9$

$= 2\left(x + \frac{5}{2}\right)^2 - \frac{25}{2} + 9$

$= \boxed{2\left(x + \frac{5}{2}\right)^2 - \frac{7}{2}}$

3(b) $\frac{1}{3}x^2 + 2x - 1 = \frac{1}{3}\left[x^2 + 6x\right] - 1$

$= \frac{1}{3}\left[(x+3)^2 - 3^2\right] - 1$

$= \frac{1}{3}(x+3)^2 - 3 - 1$

$= \boxed{\frac{1}{3}(x+3)^2 - 4}$

Section 2 — Sketching $y = a(x-h)^2 + k$ (Questions 4–6)

4(a) $y = -2\left(x^2 - \frac{7}{2}x - 2\right)$

$= -2\left[\left(x - \frac{7}{4}\right)^2 - \left(-\frac{7}{4}\right)^2 - 2\right]$

$= -2\left[\left(x - \frac{7}{4}\right)^2 - \frac{81}{16}\right]$

$= \boxed{-2\left(x - \frac{7}{4}\right)^2 + \frac{81}{8}}$

Coefficient of x^2 is $-2 < 0$: $\boxed{\text{maximum point } \left(\frac{7}{4}, \frac{81}{8}\right)}$

4(b) $y = 4x^2 - 24x + 31 = 4\left[x^2 - 6x\right] + 31$

$= 4\left[(x-3)^2 - 3^2\right] + 31$

$= 4(x-3)^2 - 36 + 31 = \boxed{4(x-3)^2 - 5}$

Coefficient of x^2 is $4 > 0$: $\boxed{\text{minimum point } (3, -5)}$

5 $f(x) = -\left[x^2 - 10x\right] - 4$

$= -\left[(x-5)^2 - 5^2\right] - 4$

$= -(x-5)^2 + 25 - 4 = \boxed{-(x-5)^2 + 21}$

Coefficient of x^2 is $-1 < 0$, so the turning point $(5, 21)$ is a maximum point.

Greatest value of $f(x)$ is $\boxed{21}$, when $\boxed{x = 5}$.

6 $y = -3x^2 + 6x - 5 = -3\left(x^2 - 2x + \frac{5}{3}\right)$

$= -3\left[(x-1)^2 - (-1)^2 + \frac{5}{3}\right]$

$= -3\left[(x-1)^2 + \frac{2}{3}\right]$

$= -3(x-1)^2 - 2$

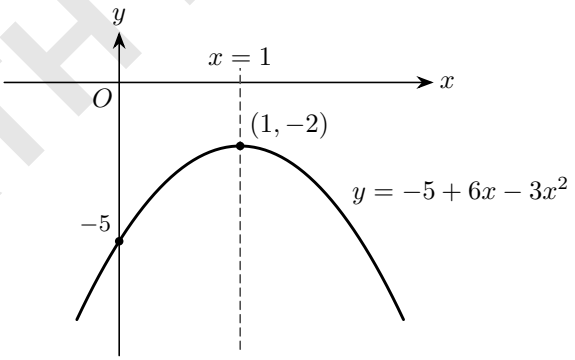
Graph Shape: coefficient of x^2 is $-3 < 0$, so n-shaped.

Turning Point: $\boxed{\text{maximum point } (1, -2)}$

Line of Symmetry: $x = 1$

y-intercept: sub $x = 0$: $y = -5$, so $(0, -5)$.

The greatest value -2 is below 0, so the whole curve lies below the x-axis: it has no x-intercepts.



Section 3 — Sketching $y = a(x-m)(x-n)$ (Questions 7–8)

7(a) $y = 2(2x+3)(x-1)$

Graph Shape: coefficient of x^2 would be $4 > 0$ after expansion, so U-shaped.

x-intercepts: sub $y = 0$: $x = -\frac{3}{2}$ or $x = 1$.

Line of Symmetry: $x = \dfrac{-1.5 + 1}{2} \Rightarrow x = -0.25$

Turning Point: sub $x = -0.25$:

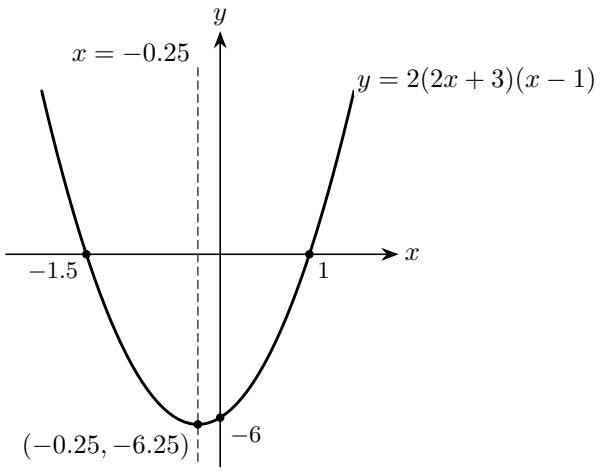
$y = 2(2.5)(-1.25) = -6.25$;

$\boxed{\text{min. point } (-0.25, -6.25)}$

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y-intercept: sub $x = 0$: $y = 2(3)(-1) = -6$.



7(b) $y = (2x + 1)(5 - x)$

Graph Shape: coefficient of x^2 would be $-2 < 0$ after expansion, so n-shaped.

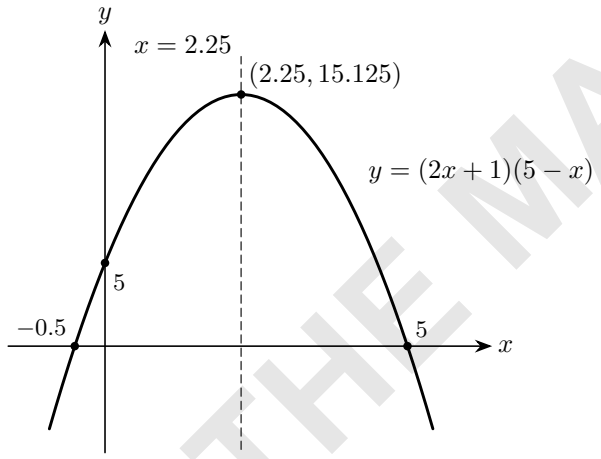
x-intercepts: sub $y = 0$: $x = -\frac{1}{2}$ or $x = 5$.

Line of Symmetry: $x = \dfrac{-0.5 + 5}{2} \Rightarrow x = 2.25$

Turning Point: sub $x = 2.25$: $y = (5.5)(2.75) = 15.125$;

max. point $(2.25, 15.125)$

y-intercept: sub $x = 0$: $y = (1)(5) = 5$.



8(a) $y = 5x - 2x^2 = x(5 - 2x)$

Graph Shape: coefficient of x^2 is $-2 < 0$, so n-shaped.

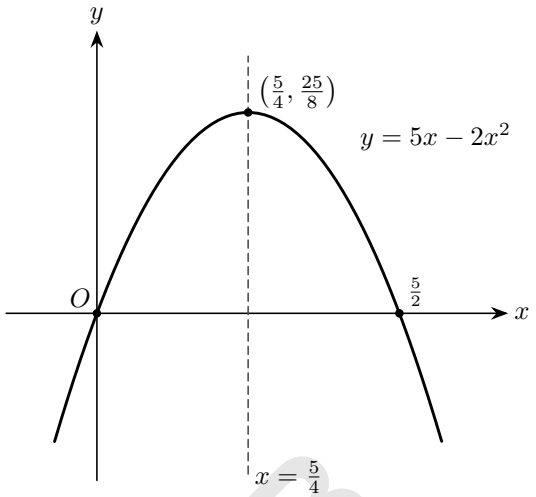
x-intercepts: sub $y = 0$: $x = 0$ or $x = \frac{5}{2}$.

Line of Symmetry: $x = \dfrac{0 + \frac{5}{2}}{2} \Rightarrow x = \frac{5}{4}$

Turning Point: sub $x = \frac{5}{4}$: $y = \frac{5}{4}\left(5 - \frac{5}{2}\right) = \frac{25}{8}$;

max. point $\left(\frac{5}{4}, \frac{25}{8}\right)$

y-intercept: sub $x = 0$: $y = 0$, so the curve passes through the origin.



8(b) $y = 16 - x^2 = (4 + x)(4 - x)$

Graph Shape: coefficient of x^2 is $-1 < 0$, so n-shaped.

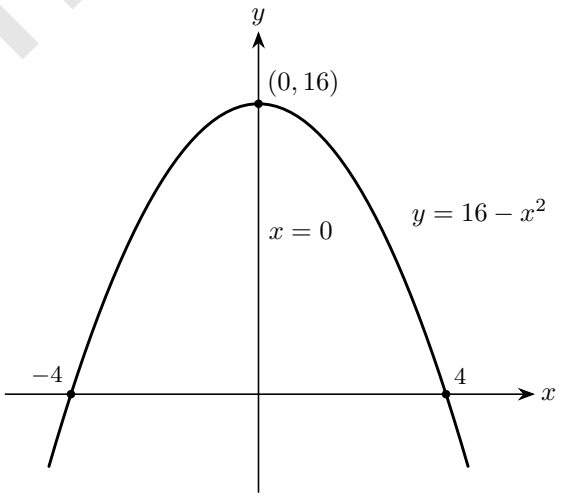
x-intercepts: sub $y = 0$: $x = -4$ or $x = 4$.

Line of Symmetry: $x = \dfrac{-4 + 4}{2} \Rightarrow x = 0$ (the y-axis)

Turning Point: sub $x = 0$: $y = 16$;

max. point $(0, 16)$

y-intercept: $y = 16$, at the maximum point.



Section 4 — Maximum and minimum values (Questions 9–12)

9 Complete the square first:

$$\begin{aligned} 27 + 24x + 4x^2 &= 4(x^2 + 6x) + 27 \\ &= 4\left[(x + 3)^2 - 3^2\right] + 27 \\ &= 4(x + 3)^2 - 9 \end{aligned}$$

Since $(x + 3)^2 \geq 0$ for all values of x

$4(x + 3)^2 \geq 0$ (*multiply by 4*)

$4(x + 3)^2 - 9 \geq -9$ (*subtract 9*)

$\therefore\ 27 + 24x + 4x^2 \geq -9$ for all values of x. (Shown)

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10 Complete the square:

$$\begin{aligned}-x^2-8x-21 &= -(x^2+8x)-21\\ &= -\left[(x+4)^2-4^2\right]-21\\ &= -(x+4)^2-5\end{aligned}$$

Since $(x+4)^2 \geq 0$ for all values of x

$-(x+4)^2 \leq 0$ (*multiply by* -1)

$-(x+4)^2-5 \leq -5$ (*subtract 5*)

The greatest value of $-x^2-8x-21$ is -5, which is negative, so the expression is negative for all values of x.

11 Complete the square:

$$\begin{aligned}-3x^2+2x+5 &= -3\left(x^2-\tfrac{2}{3}x\right)+5\\ &= -3\left[\left(x-\tfrac{1}{3}\right)^2-\left(\tfrac{1}{3}\right)^2\right]+5\\ &= -3\left(x-\tfrac{1}{3}\right)^2+\tfrac{1}{3}+5\\ &= -3\left(x-\tfrac{1}{3}\right)^2+5\tfrac{1}{3}\end{aligned}$$

Since $\left(x-\frac{1}{3}\right)^2 \geq 0$ for all values of x

$-3\left(x-\frac{1}{3}\right)^2 \leq 0$ (*multiply by* -3)

$-3\left(x-\frac{1}{3}\right)^2+5\frac{1}{3} \leq 5\frac{1}{3}$ (*add* $5\frac{1}{3}$)

$\therefore$ the greatest value is $\boxed{5\frac{1}{3}}$ (when $x=\frac{1}{3}$).

12 Complete the square:

$$\begin{aligned}x^2+8x+20 &= (x+4)^2-4^2+20\\ &= (x+4)^2+4\end{aligned}$$

Since $(x+4)^2 \geq 0$ for all values of x

$(x+4)^2+4 \geq 4$ (*add 4*)

The least value of $x^2+8x+20$ is $\boxed{4}$ (when $x=-4$).

So $x^2+8x+20 \geq 4 > 0$. Its reciprocal is therefore positive, and it is largest when $x^2+8x+20$ is smallest:

$$\boxed{0 < \frac{1}{x^2+8x+20} \leq \frac{1}{4}}$$

Section 5 — Finding the equation of a curve (Questions 13–15)

13 The turning point is $(-3,2)$, so the equation is $y=a(x+3)^2+2$.
Sub $(0,-7)$ into the equation:

$-7 = a(0+3)^2+2$

$9a = -9$

$a = -1$

$y = -(x+3)^2+2$

$y = -(x^2+6x+9)+2$

$\boxed{y = -x^2-6x-7}$

14 Line of symmetry: $\dfrac{-2+k}{2}=1$

$-2+k=2$

$k=4$

Equation: $y=a(x+2)(x-4)$
Sub $(0,16)$ into the equation:

$16 = a(0+2)(0-4)$

$-8a = 16$

$a = -2$

$\therefore\ y = -2(x+2)(x-4)$

$y = -2(x^2-2x-8)$

$y = -2x^2+4x+16$

$\boxed{k=4,\ a=-2,\ b=4,\ c=16}$

15(i) The turning point is $(3,5)$, so the equation is $y=a(x-3)^2+5$.
Sub $(1,13)$ into the equation:

$13 = a(1-3)^2+5$

$4a = 8$

$a = 2$

$\boxed{y=2(x-3)^2+5}$, i.e. $y=2x^2-12x+23$

15(ii) $y = -\frac{1}{2}x^2-x+4$

$= -\frac{1}{2}(x^2+2x-8)$

$= -\frac{1}{2}\left[(x+1)^2-1^2-8\right]$

$= -\frac{1}{2}\left[(x+1)^2-9\right]$

$\boxed{y = -\frac{1}{2}(x+1)^2+4\frac{1}{2}}$

15(iii) First curve:

Since $(x-3)^2 \geq 0$ for all values of x

$2(x-3)^2+5 \geq 5$

so its least value is 5.
Second curve:

Since $(x+1)^2 \geq 0$ for all values of x

$-\frac{1}{2}(x+1)^2 \leq 0$

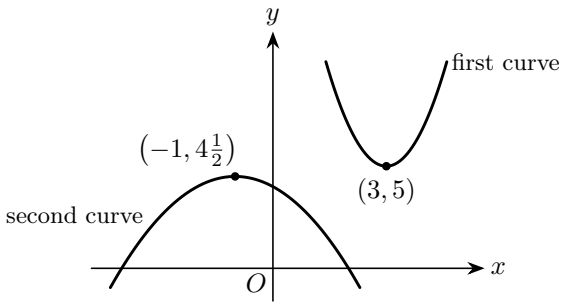
$-\frac{1}{2}(x+1)^2+4\frac{1}{2} \leq 4\frac{1}{2}$

so its greatest value is $4\frac{1}{2}$.
The least value of the first curve (5) is greater than the greatest value of the second curve ($4\frac{1}{2}$). So the first

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curve lies entirely above the second, and the two curves do not meet.



Section 6 — Application Questions (Questions 16–21)

16(i) Sub $x = 0$: $H = 2.2$. The ball is hit from a height of $\boxed{2.2\,\text{m}}$ above the ground.

16(ii)

$$\begin{aligned} H &= -0.05x^2 + 0.6x + 2.2 \\ &= -0.05\left(x^2 - 12x - 44\right) \\ &= -0.05\left[(x-6)^2 - 6^2 - 44\right] \\ &= -0.05\left[(x-6)^2 - 80\right] \end{aligned}$$

$\boxed{-0.05(x-6)^2 + 4}$

$(a = -0.05,\ h = 6,\ k = 4.)$

16(iii) Maximum point = (6, 4).

Greatest height = $\boxed{4\,\text{m}}$

16(iv) Sub $x = 9$:

$$H = -0.05(9-6)^2 + 4 = 3.55$$

When the ball reaches the net it is 3.55 m above the ground. Since $3.55 > 2.43$,

$\boxed{\text{the ball passes over the net}}$.

17(i) Complete the square:

$$\begin{aligned} y &= -5t^2 + 30t + c \\ &= -5\left(t^2 - 6t\right) + c \\ &= -5\left[(t-3)^2 - 3^2\right] + c \\ &= -5(t-3)^2 + 45 + c \end{aligned}$$

Since $(t-3)^2 \geq 0$, $-5(t-3)^2 \leq 0$, so $y \leq 45 + c$: the greatest height is $45 + c$ m.

The firework does not reach 60 m when

$$45 + c < 60 \quad \Rightarrow \quad c < 15.$$

Since c is the height of the platform, $c > 0$:

$\boxed{0 < c < 15}$

17(ii)

$$\begin{aligned} y &= -5t^2 + 30t + 10 \\ &= -5\left(t^2 - 6t - 2\right) \\ &= -5\left[(t-3)^2 - 3^2 - 2\right] \\ &= \boxed{-5(t-3)^2 + 55} \end{aligned}$$

Maximum point $\boxed{(3,\ 55)}$

17(iii) From (i), when $c = 10$ (which lies between 0 and 15) the firework does not reach 60 m. This agrees with (ii): its greatest height is 55 m, which is less than 60 m.

18(i) The lowest point is halfway between the pylons: minimum point = (60, 14).

$$y = a(x-60)^2 + 14$$

Sub $x = 0$, $y = 30$ (*top of pylon A*):

$$30 = a(0-60)^2 + 14 \ \Rightarrow \ a = \tfrac{16}{3600} = \tfrac{1}{225}$$

$\boxed{y = \frac{1}{225}(x-60)^2 + 14}$

18(ii) Sub $y = 18$:

$$\begin{aligned} 18 &= \tfrac{1}{225}(x-60)^2 + 14 \\ (x-60)^2 &= 900 \\ x - 60 &= \pm 30 \end{aligned}$$

The power line is 18 m above the ground at $\boxed{30\,\text{m or } 90\,\text{m}}$ from pylon A.

19(i)

$$\begin{aligned} y &= -0.48\left(x^2 - 6x\right) \\ &= -0.48\left[(x-3)^2 - 3^2\right] \\ &= \boxed{-0.48(x-3)^2 + 4.32} \end{aligned}$$

19(ii) Highest point $\boxed{(3,\ 4.32)}$

19(iii) Take the cross section of the boat as a rectangle. Width of the arch: sub $y = 0$: $-0.48x(x-6) = 0$, so $x = 0$ or $x = 6$.

The boat has the most room in the middle of the arch. Its left side is then at

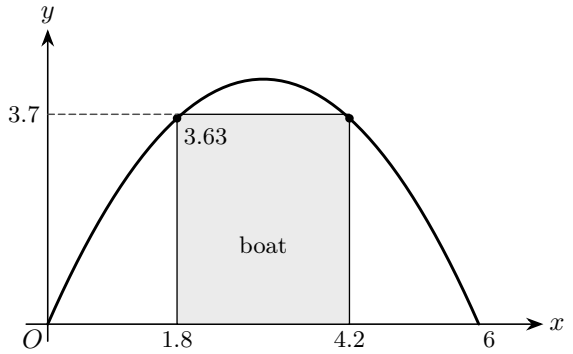
$$x = \tfrac{6-2.4}{2} = 1.8.$$

Sub $x = 1.8$:

$$y = -0.48(1.8-3)^2 + 4.32 = 3.6288$$

At the sides of the boat the arch is only 3.63 m above the water, and $3.63 < 3.7$:

$\boxed{\text{the boat cannot pass under the bridge}}$.



20(i) When $x = 0$, $y = -6000$: if no money came in from tickets, the concert would lose $\boxed{\$6000}$, the cost of hiring the hall.

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20(ii) The concert breaks even when $y = 0$:

$$-20x^2 + 1000x - 6000 = 0$$
$$x^2 - 50x + 300 = 0$$
$$x = \frac{50 \pm \sqrt{50^2 - 4(1)(300)}}{2}$$

$x = 6.97$ or $x = 43.0$ (3 s.f.)
The curve is n-shaped, so the money raised is positive between the two roots. The concert starts to raise money once $x > 6.97$; since the price is a whole number of dollars, the lowest price is $\boxed{\$7}$.

20(iii)

$$\begin{aligned} y &= -20\left(x^2 - 50x + 300\right) \\ &= -20\left[(x-25)^2 - 25^2 + 300\right] \\ &= -20(x-25)^2 + 6500 \end{aligned}$$

Maximum point (25, 6500): a ticket price of $\boxed{\$25}$ raises the most money, $\boxed{\$6500}$.

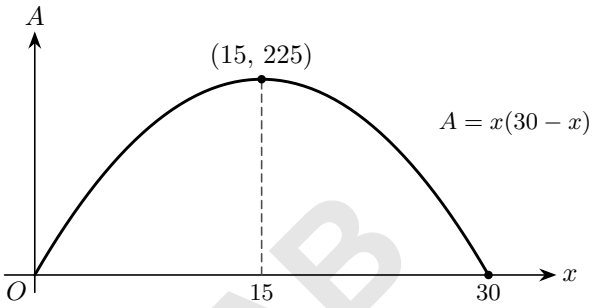
21(i) Perimeter = 60, so the length is $\frac{60-2x}{2} = 30 - x$.
$\boxed{A = x(30 - x)}$

21(ii) Sub $A = 0$: $x(30 - x) = 0$, so
$\boxed{x = 0 \text{ and } x = 30}$.

21(iii)

$$\begin{aligned} A &= -\left(x^2 - 30x\right) \\ &= -\left[(x-15)^2 - 15^2\right] \\ &= -(x-15)^2 + 225 \end{aligned}$$

Maximum point (15, 225). Greatest area = $\boxed{225\,\text{m}^2}$



21(iv) The greatest area occurs only at $x = 15$. Then the width is 15 m and the length is $30 - 15 = 15$ m: the two are equal, so the pen is a square. (Shown)

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