Equations and Inequalities

Simultaneous Equations, Quadratic Inequalities and the Discriminant

Secondary 3 — G3 Additional Mathematics

Notes

Solving a Linear and a Non-linear Equation Together

Step 1: Take the **linear equation** and make x or y the subject. Choose the one that avoids fractions if you can.

Step 2: Substitute this expression into the **non-linear equation**. You now have a quadratic equation in one unknown.

Step 3: Solve the quadratic equation. Put each answer back into the equation from Step 1 to find the matching value of the other unknown.

Each pair of answers (x, y) is a point where the straight line meets the curve. A line usually meets the curve at two points, so there are usually two pairs of answers. Always write each value of x together with its own value of y.

In a word problem, give each unknown quantity its own letter. Use the information in the question to write two equations: usually one is linear and the other is not. Solve them by substitution, as above. Then check every answer against the problem: a length or an age must be positive, and a digit must be a whole number from 0 to 9.

A quadratic inequality asks for the values of x that make a quadratic expression positive or negative. A quick sketch of the curve shows the answer at a glance.

Steps for solving a quadratic inequality

Step 1: Bring all the terms to one side, so that the inequality compares $ax^2 + bx + c$ with **0**.

Step 2: If a is negative, multiply both sides by -1 so that a becomes **positive**. Multiplying by a negative number **turns the inequality sign round**.

Step 3: Solve $ax^2 + bx + c = 0$. The roots are where the curve cuts the x-axis.

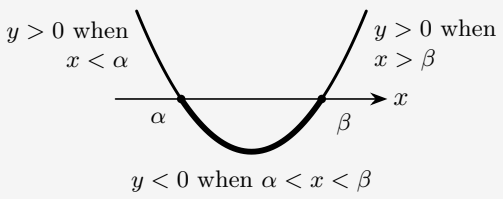
Step 4: **Sketch** the U-shaped curve $y = ax^2 + bx + c$, marking the roots on the x-axis.

Step 5: Read the answer from the sketch: "< 0" is the part of the curve below the x-axis and "> 0" is the part above it.

How to read the sketch. When $a > 0$ the curve is U-shaped. Suppose it cuts the x-axis at $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

- The curve is **below** the x-axis ($y < 0$) only **between** the roots: $\alpha < x < \beta$. This is one range of values.
- The curve is **above** the x-axis ($y > 0$) **outside** the roots: $x < \alpha$ or $x > \beta$. This is two ranges of values.

For $\leq$ or $\geq$, the roots are included as well. In the sketches in the solutions, the thick part of the curve shows the values of x we want.



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Solving two inequalities together. Solve each inequality on its own. Show both answers on one number line: the values of x we want are where the two answers **overlap** (the part shaded with lines). A filled dot • means the end value is included ($\leq$ or $\geq$); an open dot ∘ means it is not included ($<$ or $>$). A double inequality such as $p \leq f(x) < q$ is really two inequalities, $f(x) \geq p$ **and** $f(x) < q$, and both must be true.

Unknown coefficients. When the solution of an inequality is given and its coefficients are to be found, work backwards: start from the solution, write down the factors, expand, and then match with the inequality in the question.

- A solution between two values, $\alpha < x < \beta$, comes from $(x-\alpha)(x-\beta) < 0$.
- A solution outside them, $x < \alpha$ or $x > \beta$, comes from $(x-\alpha)(x-\beta) > 0$.

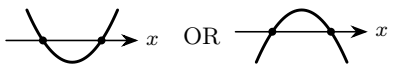
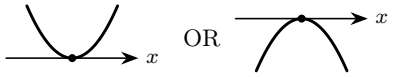
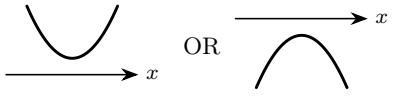
Use $\leq$ or $\geq$ when the end values are included. If the coefficient of x^2 in the question is not 1, multiply both sides by the number that matches it; multiplying by a negative number turns the inequality sign round.

The Discriminant

For the quadratic equation $ax^2 + bx + c = 0$, the **discriminant** is

$$D = b^2 - 4ac$$

It is the part under the square root sign in the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Its sign tells us how many real roots the equation has. The real roots are the x-coordinates of the points where the curve $y = ax^2 + bx + c$ meets the x-axis.

$\boldsymbol{b^2 - 4ac > 0}$	**two real, different roots** (the curve cuts the x-axis)	
$\boldsymbol{b^2 - 4ac = 0}$	**two equal real roots** (the curve touches the x-axis)	
$\boldsymbol{b^2 - 4ac < 0}$	**no real roots** (the curve does not meet the x-axis)	

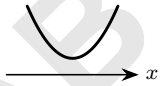
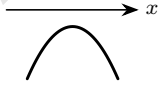
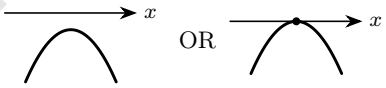
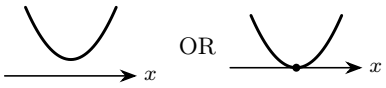
- "The equation has **real roots**" means $b^2 - 4ac \geq 0$: the roots are either different or equal.
- Always rearrange the equation into the form $ax^2 + bx + c = 0$ first, and then read off a, b and c.
- If a contains the unknown, as in $(k-3)x^2 + \ldots$, look separately at the value that makes $a = 0$: the equation is then no longer quadratic.

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For all real values of a constant. To show that a condition on $b^2 - 4ac$ holds for all real values of a constant k (for example, that an equation has real roots, or no real roots), write $b^2 - 4ac$ in terms of k. Then complete the square, or write it as a perfect square, and use the fact that a square is never negative.

An expression $ax^2 + bx + c$ is **always positive** when the curve $y = ax^2 + bx + c$ lies completely above the x-axis, and **always negative** when the curve lies completely below it. In both cases the curve does not meet the x-axis, so $b^2 - 4ac < 0$; the sign of a then tells us whether the curve is above or below the axis.

For all real values of x, $ax^2 + bx + c$ is	**Conditions**		**Graph**
	Discriminant $b^2 - 4ac$	**Coefficient of x^2, a**	
ALWAYS POSITIVE $y > 0$, that is, $ax^2 + bx + c > 0$	$\boldsymbol{b^2 - 4ac < 0}$	$\boldsymbol{a > 0}$	U-shaped curve, completely above the x-axis 
ALWAYS NEGATIVE $y < 0$, that is, $ax^2 + bx + c < 0$	$\boldsymbol{b^2 - 4ac < 0}$	$\boldsymbol{a < 0}$	n-shaped curve, completely below the x-axis 
Never positive (negative or zero)	$\boldsymbol{b^2 - 4ac \leq 0}$	$\boldsymbol{a < 0}$	
Never negative (positive or zero)	$\boldsymbol{b^2 - 4ac \leq 0}$	$\boldsymbol{a > 0}$	

- Both conditions must hold. Solve the inequality for $b^2 - 4ac$, then keep only the values that also satisfy the condition on a.
- To show that an expression is always positive or always negative, either complete the square, or show that both conditions in the table hold.

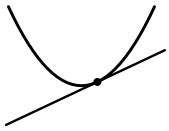
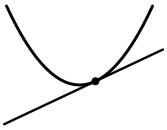
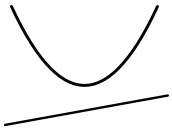
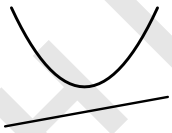
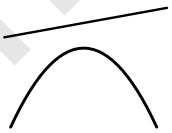
Steps for a Line and a Curve

Step 1: Put the two equations equal to each other (substitute the line into the curve), then bring every term to one side to get a quadratic equation $ax^2 + bx + c = 0$.

Step 2: Use the discriminant $b^2 - 4ac$ with the condition that describes how the line and the curve meet.

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How the line and the curve meet	Picture	Condition
The line cuts the curve at two different points		$b^2 - 4ac > 0$
The line is a tangent to the curve **The line touches the curve** **The line meets the curve at exactly one point, and the equation is a quadratic with equal roots**		$b^2 - 4ac = 0$
The line meets (intersects) the curve (one point or two points are both allowed)	 or 	$b^2 - 4ac \geq 0$
The line does not meet the curve		$b^2 - 4ac < 0$
The curve lies entirely above the line		$b^2 - 4ac < 0$ and $a > 0$
The curve lies entirely below the line		$b^2 - 4ac < 0$ and $a < 0$

In the last two rows, a is the coefficient of x^2 in the equation of the curve. Only a U-shaped curve ($a > 0$) can stay above a line, and only an n-shaped curve ($a < 0$) can stay below one.

If the coefficient of x^2 contains the unknown, for example $(k+1)x^2$, look at the value that makes it zero (here $k = -1$). For that value the equation is linear, not quadratic, so the discriminant cannot be used: solve the linear equation on its own. It usually has exactly one root, and then the line meets the curve at one point only. Such a line crosses the curve there instead of just touching it, so it is not a tangent.

Steps for an Inequality that Holds for All Real Values of x

Step 1: Move every term to one side, so that the inequality reads $ax^2 + bx + c > 0$ (or < 0, ≥ 0, ≤ 0) for all real values of x.

Step 2: Use the conditions for always positive and always negative given above: > 0 for all x means "always positive", < 0 means "always negative", ≥ 0 means "never negative" and ≤ 0 means "never positive".

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Exercises

Unless a question says otherwise, give non-exact answers correct to 3 significant figures.

Section 1: Simultaneous Equations: One Linear, One Non-linear

1. The line $x - 3y = 2$ meets the curve $xy = 2x + 11$ at the points A and B. Find the coordinates of A and of B.

2. Find the coordinates of the points where the line $3x + y = 5$ meets the curve $x^2 + y^2 - 5x = 10$.

3. Solve each pair of simultaneous equations.

(a) $\dfrac{3}{x} + \dfrac{1}{y} = 7$, $\dfrac{1}{xy} = 2$

(b) $x + 2y = 4$, $x^2 - 6xy - 7 = 0$

(c) $x + y + 4 = 0$, $(2x - y)(3x - y) = 8$

Section 2: Problems Leading to Simultaneous Equations

4. Two positive numbers differ by 3, and the sum of their squares is 117. Find the two numbers.

5. A two-digit number is equal to 4 times the sum of its digits. It is also equal to 3 times the product of its digits. Find the number.

6. It is given that $(2, p)$ is a solution of the simultaneous equations

$$x^2 + 3y^2 = 7, \qquad p^2x + 3y = -1.$$

(i) Find the value of p.
(ii) Find the other solution of the pair of equations.

7. The hypotenuse of a right-angled triangle is 17 cm long, and the perimeter of the triangle is 40 cm. Find the lengths of the other two sides.

Section 3: Quadratic Inequalities

8. Solve each inequality.

(a) $(3x + 2)(x - 4) \geq 0$ (b) $(2x - 1)^2 - 2\frac{1}{4} \geq 0$

9. For each of the following, find the range of values of x.

(a) $(x + 2)(x - 3) > 6$ (b) $2x^2 + 9x \leq 2(x + 2)$

10. Solve each inequality.

(a) $x^2 + 6x \leq 0$ (b) $(3 - x)^2 \geq 34 - 6x$

Section 4: Combined Inequalities and Unknown Coefficients

11. Find the range of values of x for which both of these inequalities are true:

$$5 - 2x > 2, \qquad 2x^2 \leq 5x - 2.$$

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12. In each part, find the value of a and of b.

(a) The solution of $-x^2 + ax - b \geq 0$ is $-1 \leq x \leq 5$.

(b) The solution of $3x^2 + ax < b$ is $-2 < x < 3$.

(c) The solution of $ax - 2x^2 < b$ is $x < -1$ or $x > 1\frac{1}{2}$.

13. Find the range of values of x for which

$$x(x+2) < 2x^2 + 3x - 2 < 3.$$

Section 5: The Discriminant and Its Applications

14. Find the values of k for which the equation

(a) $3x^2 + 2kx^2 - 6x = kx - 3$ has real and distinct roots,

(b) $x^2 + 4x + 1 = k(x^2 + 1)$ has exactly one solution.

15. Explain why the equation $3x^2 - 4kx + k^2 - 2 = 0$, where k is a constant, has real roots for every real value of k.

16. Show that the equation $x^2 - 2kx + 2k^2 - 2k + 3 = 0$ has no real roots for all real values of k.

17. Show that the expression $-2x^2 + 3x - 4$ is negative for all real values of x.

18. Show that $y = x^2 + 5k^2 + 2 + 4kx$ is positive for all real values of x and k.

19. Find the range of values of p for which $2px^2 + 5x + 8p$ is negative for all real values of x.

20. Find the range of values of p for which $x^2 - 6x + p(2 - x) + 9$ is never negative.

21. Find the range of values of m for which the line $y = mx - 3$ does not meet the curve $y = x^2 + x + 1$.

22. Find the range of values of c for which the line $cx - y = 2$ meets the curve $x^2 + xy = 4$ at two different points. Hence state the value of c for which the line is a tangent to the curve.

23. Find the range of values of p for which the curve $y = px^2 + 6x + p$ lies entirely above the line $y = 8$.

24. Show that the line $y + px = 2p$ meets the curve $y = (p+3)x^2 + px - 3$ at two different points for all real values of p, where $p \neq -3$.

25. Find the range of values of m for which the curve $y = -x^2 + 4x - 1$ lies entirely below the line $y = mx + m + 3$.

26. Find the range of values of p for which $px^2 + 8x + p \leq 6$ for all real values of x.

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Solutions to Exercises

Section 1 — Simultaneous equations (Questions 1–3)

1 $x - 3y = 2 \quad —(1)$

$xy = 2x + 11 \quad —(2)$

From (1): $x = 3y + 2 \quad —(3)$

Sub (3) into (2):

$$(3y+2)y = 2(3y+2) + 11$$
$$3y^2 + 2y = 6y + 15$$
$$3y^2 - 4y - 15 = 0$$
$$(3y+5)(y-3) = 0$$
$$y = -\tfrac{5}{3} \quad \text{or} \quad y = 3$$

When $y = -\frac{5}{3}$, $x = 3\left(-\frac{5}{3}\right) + 2 = -3$. When $y = 3$, $x = 3(3) + 2 = 11$.

$\boxed{A(11,\ 3),\ B\left(-3,\ -1\tfrac{2}{3}\right)}$ (or the other way round)

2 $3x + y = 5 \quad —(1)$

$x^2 + y^2 - 5x = 10 \quad —(2)$

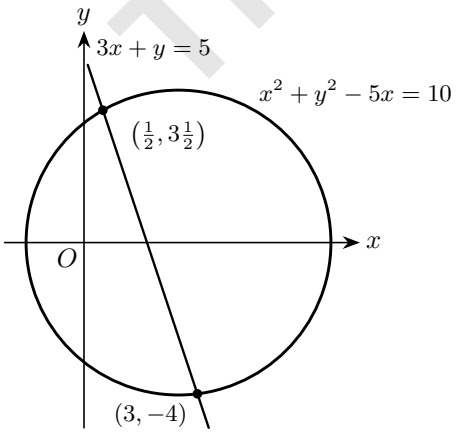
From (1): $y = 5 - 3x \quad —(3)$

Sub (3) into (2):

$$x^2 + (5-3x)^2 - 5x = 10$$
$$x^2 + 25 - 30x + 9x^2 - 5x - 10 = 0$$
$$10x^2 - 35x + 15 = 0$$
$$2x^2 - 7x + 3 = 0$$
$$(2x-1)(x-3) = 0$$
$$x = \tfrac{1}{2} \quad \text{or} \quad x = 3$$

When $x = \frac{1}{2}$, $y = 5 - 3\left(\frac{1}{2}\right) = \frac{7}{2}$. When $x = 3$, $y = 5 - 3(3) = -4$.

Coordinates: $\boxed{\left(\tfrac{1}{2},\ 3\tfrac{1}{2}\right) \text{ and } (3,\ -4)}$



3(a) $\dfrac{3}{x} + \dfrac{1}{y} = 7 \quad —(1)$

$\dfrac{1}{xy} = 2 \quad —(2)$

From (2): $2xy = 1 \quad —(3)$

Multiply (1) by xy, then use $xy = \frac{1}{2}$:

$$3y + x = 7xy = \tfrac{7}{2}$$
$$2x + 6y = 7$$
$$x = \frac{7 - 6y}{2} \quad —(4)$$

Sub (4) into (3):

$$2\left(\frac{7-6y}{2}\right)y = 1$$
$$(7-6y)y = 1$$
$$6y^2 - 7y + 1 = 0$$
$$(6y-1)(y-1) = 0$$
$$y = \tfrac{1}{6} \quad \text{or} \quad y = 1$$

When $y = \frac{1}{6}$, $x = \dfrac{7 - 6\left(\frac{1}{6}\right)}{2} = 3$.

When $y = 1$, $x = \dfrac{7 - 6(1)}{2} = \dfrac{1}{2}$.

$\boxed{x = 3,\ y = \tfrac{1}{6} \ \text{ or } \ x = \tfrac{1}{2},\ y = 1}$

3(b) $x + 2y = 4 \quad —(1)$

$x^2 - 6xy - 7 = 0 \quad —(2)$

From (1): $x = 4 - 2y \quad —(3)$

Sub (3) into (2):

$$(4-2y)^2 - 6(4-2y)y - 7 = 0$$
$$16 - 16y + 4y^2 - 24y + 12y^2 - 7 = 0$$
$$16y^2 - 40y + 9 = 0$$
$$(4y-9)(4y-1) = 0$$
$$y = \tfrac{9}{4} \quad \text{or} \quad y = \tfrac{1}{4}$$

When $y = \frac{9}{4}$, $x = 4 - 2\left(\frac{9}{4}\right) = -\frac{1}{2}$.

When $y = \frac{1}{4}$, $x = 4 - 2\left(\frac{1}{4}\right) = \frac{7}{2}$.

$\boxed{x = -\tfrac{1}{2},\ y = 2\tfrac{1}{4} \ \text{ or } \ x = 3\tfrac{1}{2},\ y = \tfrac{1}{4}}$

3(c) $x + y + 4 = 0 \quad —(1)$

$(2x - y)(3x - y) = 8 \quad —(2)$

From (1): $y = -x - 4 \quad —(3)$

Sub (3) into (2):

$$(2x + x + 4)(3x + x + 4) = 8$$
$$(3x+4)(4x+4) = 8$$
$$12x^2 + 12x + 16x + 16 - 8 = 0$$
$$12x^2 + 28x + 8 = 0$$
$$3x^2 + 7x + 2 = 0$$
$$(3x+1)(x+2) = 0$$
$$x = -\tfrac{1}{3} \quad \text{or} \quad x = -2$$

When $x = -\frac{1}{3}$, $y = \frac{1}{3} - 4 = -\frac{11}{3}$. When $x = -2$, $y = 2 - 4 = -2$.

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$\boxed{x = -\frac{1}{3},\ y = -3\frac{2}{3} \text{ or } x = -2,\ y = -2}$

Section 2 — Problems leading to simultaneous equations (Questions 4–7)

4 Let the larger number be x and the smaller number be y.

$$x - y = 3 \quad —(1)$$
$$x^2 + y^2 = 117 \quad —(2)$$

From (1): $x = y + 3$ —(3)
Sub (3) into (2):

$$(y+3)^2 + y^2 = 117$$
$$y^2 + 6y + 9 + y^2 - 117 = 0$$
$$2y^2 + 6y - 108 = 0$$
$$y^2 + 3y - 54 = 0$$
$$(y+9)(y-6) = 0$$
$$y = -9 \text{ (rejected) or } y = 6$$

The numbers are positive, so $y \neq -9$. When $y = 6$, $x = 6 + 3 = 9$.
$\boxed{\text{The numbers are 9 and 6.}}$

5 Let the tens digit be t and the units digit be u. The number is $10t + u$.

$$10t + u = 4(t+u) \quad —(1)$$
$$10t + u = 3tu \quad —(2)$$

From (1): $10t + u = 4t + 4u$, $6t = 3u$, $u = 2t$ —(3)
Sub (3) into (2):

$$10t + 2t = 3t(2t)$$
$$12t = 6t^2$$
$$6t^2 - 12t = 0$$
$$6t(t-2) = 0$$
$$t = 0 \text{ (rejected) or } t = 2$$

The tens digit of a two-digit number cannot be 0, so $t = 2$ and $u = 2(2) = 4$.
Check: $4(2+4) = 24$ and $3(2)(4) = 24$.
The number is $\boxed{24}$

6(i) Sub $x = 2$, $y = p$ into both equations.
First equation:

$$4 + 3p^2 = 7$$
$$p^2 = 1$$
$$p = 1 \text{ or } p = -1$$

Second equation:

$$2p^2 + 3p = -1$$
$$2p^2 + 3p + 1 = 0$$
$$(2p+1)(p+1) = 0$$
$$p = -\frac{1}{2} \text{ or } p = -1$$

Both equations hold only when $\boxed{p = -1}$

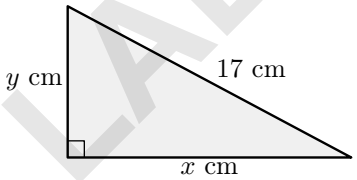
6(ii) With $p = -1$, $p^2 = 1$: $x^2 + 3y^2 = 7$ —(1)
$x + 3y = -1$ —(2)

From (2): $x = -1 - 3y$ —(3)
Sub (3) into (1):

$$(-1-3y)^2 + 3y^2 = 7$$
$$1 + 6y + 9y^2 + 3y^2 - 7 = 0$$
$$12y^2 + 6y - 6 = 0$$
$$2y^2 + y - 1 = 0$$
$$(2y-1)(y+1) = 0$$
$$y = \tfrac{1}{2} \text{ or } y = -1$$

$y = -1$ gives $x = 2$, the given solution $(2, -1)$.
When $y = \frac{1}{2}$, $x = -1 - 3\left(\frac{1}{2}\right) = -\frac{5}{2}$.
The other solution is $\boxed{x = -2\frac{1}{2},\ y = \frac{1}{2}}$

7



Let the other two sides be x cm and y cm.
Perimeter: $x + y + 17 = 40$. By Pythagoras' theorem, $x^2 + y^2 = 17^2$.

$$x + y = 23 \quad —(1)$$
$$x^2 + y^2 = 289 \quad —(2)$$

From (1): $x = 23 - y$ —(3)
Sub (3) into (2):

$$(23-y)^2 + y^2 = 289$$
$$529 - 46y + 2y^2 - 289 = 0$$
$$2y^2 - 46y + 240 = 0$$
$$y^2 - 23y + 120 = 0$$
$$(y-8)(y-15) = 0$$
$$y = 8 \text{ or } y = 15$$

When $y = 8$, $x = 15$; when $y = 15$, $x = 8$.
The other two sides are $\boxed{8\,\text{cm and } 15\,\text{cm}}$

Section 3 — Quadratic inequalities (Questions 8–10)

8(a) $(3x+2)(x-4) \geq 0$



$\boxed{x \leq -\frac{2}{3} \text{ or } x \geq 4}$

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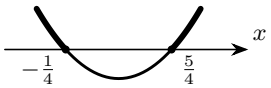
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8(b) $(2x-1)^2 - 2\frac{1}{4} \geq 0$

$4x^2 - 4x + 1 - \frac{9}{4} \geq 0$

$16x^2 - 16x - 5 \geq 0$

$(4x+1)(4x-5) \geq 0$



$\boxed{x \leq -\frac{1}{4} \text{ or } x \geq \frac{5}{4}}$

9(a) $(x+2)(x-3) > 6$

$x^2 - x - 6 - 6 > 0$

$x^2 - x - 12 > 0$

$(x-4)(x+3) > 0$



$\boxed{x < -3 \text{ or } x > 4}$

9(b) $2x^2 + 9x \leq 2(x+2)$

$2x^2 + 9x - 2x - 4 \leq 0$

$2x^2 + 7x - 4 \leq 0$

$(2x-1)(x+4) \leq 0$



$\boxed{-4 \leq x \leq \frac{1}{2}}$

10(a) $x^2 + 6x \leq 0$

$x(x+6) \leq 0$



$\boxed{-6 \leq x \leq 0}$

10(b) $(3-x)^2 \geq 34 - 6x$

$9 - 6x + x^2 - 34 + 6x \geq 0$

$x^2 - 25 \geq 0$

$(x-5)(x+5) \geq 0$



$\boxed{x \leq -5 \text{ or } x \geq 5}$

Section 4 — Combined inequalities and unknown coefficients (Questions 11–13)

11 $5 - 2x > 2 \;\Rightarrow\; -2x > -3 \;\Rightarrow\; x < \frac{3}{2}$

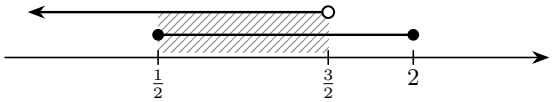
$2x^2 \leq 5x - 2$

$2x^2 - 5x + 2 \leq 0$

$(2x-1)(x-2) \leq 0$

$\frac{1}{2} \leq x \leq 2$





$\boxed{\frac{1}{2} \leq x < \frac{3}{2}}$

12(a) $-1 \leq x \leq 5$

$(x+1)(x-5) \leq 0$

$x^2 - 4x - 5 \leq 0$

$-x^2 + 4x + 5 \geq 0$



Compare with $-x^2 + ax - b \geq 0$: $\boxed{a = 4,\ b = -5}$

12(b) $-2 < x < 3$

$(x+2)(x-3) < 0$

$x^2 - x - 6 < 0$

$3x^2 - 3x - 18 < 0$

$3x^2 - 3x < 18$



Compare with $3x^2 + ax < b$: $\boxed{a = -3,\ b = 18}$

12(c) $x < -1$ or $x > \frac{3}{2}$

$(x+1)\left(x - \frac{3}{2}\right) > 0$

$x^2 - \frac{1}{2}x - \frac{3}{2} > 0$

$-2x^2 + x + 3 < 0$

$x - 2x^2 < -3$



(Multiplying by -2 turns the sign round.)

Compare with $ax - 2x^2 < b$: $\boxed{a = 1,\ b = -3}$

13 $x(x+2) < 2x^2 + 3x - 2$ **and** $2x^2 + 3x - 2 < 3$.

$x^2 + 2x < 2x^2 + 3x - 2$

$0 < x^2 + x - 2$

$(x+2)(x-1) > 0$

$x < -2$ or $x > 1$



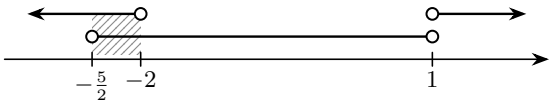
$2x^2 + 3x - 2 < 3$

$2x^2 + 3x - 5 < 0$

$(2x+5)(x-1) < 0$

$-\frac{5}{2} < x < 1$





$\boxed{-\frac{5}{2} < x < -2}$

Section 5 — The discriminant and its applications (Questions 14–26)

14(a) $3x^2 + 2kx^2 - 6x = kx - 3$

$(3+2k)x^2 - (6+k)x + 3 = 0$

Real and distinct roots, $b^2 - 4ac > 0$

$$\big(-(6+k)\big)^2 - 4(3+2k)(3) > 0$$

$$36 + 12k + k^2 - 36 - 24k > 0$$

$$k^2 - 12k > 0$$

$$k(k-12) > 0$$



$k < 0$ or $k > 12$

For the equation to be quadratic, $3 + 2k \neq 0$, so $k \neq -\frac{3}{2}$.

$\boxed{k < 0,\ k \neq -\frac{3}{2},\ \text{or } k > 12}$

14(b) $x^2 + 4x + 1 = kx^2 + k$

$(1-k)x^2 + 4x + (1-k) = 0$

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If $k \neq 1$, exactly one solution means equal roots, $b^2 - 4ac = 0$:

$$(4)^2 - 4(1-k)(1-k) = 0$$
$$16 - 4(1-k)^2 = 0$$
$$(1-k)^2 = 4$$
$$1 - k = 2 \quad \text{or} \quad 1 - k = -2$$
$$k = -1 \quad \text{or} \quad k = 3$$

If $k = 1$, the equation is $4x = 0$, which has exactly one solution, $x = 0$.

$\boxed{k = -1,\ k = 1 \text{ or } k = 3}$

15 $3x^2 - 4kx + k^2 - 2 = 0$

$$b^2 - 4ac = (-4k)^2 - 4(3)(k^2 - 2)$$
$$= 16k^2 - 12k^2 + 24$$
$$= 4k^2 + 24$$

Since $k^2 \geq 0$ for all real values of k,
$4k^2 + 24 \geq 24$, so $b^2 - 4ac > 0$.
$\therefore$ The equation has real (and distinct) roots for every real value of k.

16 $x^2 - 2kx + 2k^2 - 2k + 3 = 0$
To show: $b^2 - 4ac < 0$

$$b^2 - 4ac$$
$$= (-2k)^2 - 4(1)(2k^2 - 2k + 3)$$
$$= 4k^2 - 8k^2 + 8k - 12$$
$$= -4k^2 + 8k - 12$$
$$= -4\left(k^2 - 2k + 3\right)$$
$$= -4\left[(k-1)^2 - (-1)^2 + 3\right]$$
$$= -4\left[(k-1)^2 + 2\right]$$
$$= -4(k-1)^2 - 8$$

Since $(k-1)^2 \geq 0$ for all real values of k,

$$-4(k-1)^2 \leq 0$$
$$-4(k-1)^2 - 8 \leq -8$$

so $b^2 - 4ac < 0$.
$\therefore$ The equation has no real roots for all real values of k. (Shown)

17 To show that $b^2 - 4ac < 0$ and $a < 0$:

$$b^2 - 4ac = (3)^2 - 4(-2)(-4)$$
$$= 9 - 32$$
$$= -23$$

Since $b^2 - 4ac = -23 < 0$ and $a = -2 < 0$, the expression is negative for all real values of x. (Shown)

18 $y = x^2 + 4kx + (5k^2 + 2)$
To show that $b^2 - 4ac < 0$ and $a > 0$:

$$b^2 - 4ac = (4k)^2 - 4(1)(5k^2 + 2)$$
$$= 16k^2 - 20k^2 - 8$$
$$= -4k^2 - 8$$

Since $k^2 \geq 0$ for all real values of k,
$-4k^2 \leq 0$, $-4k^2 - 8 \leq -8$, so $b^2 - 4ac < 0$.
Also $a = 1 > 0$.
$\therefore$ y is positive for all real values of x and k. (Shown)

19 Conditions for always negative:

$$b^2 - 4ac < 0 \quad \text{and} \quad a < 0$$
$$(5)^2 - 4(2p)(8p) < 0 \quad \text{and} \quad 2p < 0$$
$$25 - 64p^2 < 0 \quad \text{and} \quad p < 0$$
$$64p^2 - 25 > 0$$
$$(8p + 5)(8p - 5) > 0$$



$p < -\frac{5}{8}$ or $p > \frac{5}{8}$ (rejected)
Since $p < 0$ as well, $\boxed{p < -\frac{5}{8}}$

20 $x^2 - 6x + p(2 - x) + 9 = x^2 - (6 + p)x + (9 + 2p)$
Never negative: $b^2 - 4ac \leq 0$ and $a > 0$ ($a = 1 > 0$, true)

$$\left(-(6+p)\right)^2 - 4(1)(9 + 2p) \leq 0$$
$$36 + 12p + p^2 - 36 - 8p \leq 0$$
$$p^2 + 4p \leq 0$$
$$p(p + 4) \leq 0$$



$\boxed{-4 \leq p \leq 0}$

21 $x^2 + x + 1 = mx - 3$

$$x^2 + x - mx + 4 = 0$$
$$x^2 + (1 - m)x + 4 = 0$$

Does not meet: $b^2 - 4ac < 0$

$$(1-m)^2 - 4(1)(4) < 0$$
$$m^2 - 2m + 1 - 16 < 0$$
$$m^2 - 2m - 15 < 0$$
$$(m - 5)(m + 3) < 0$$



$\boxed{-3 < m < 5}$

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22 $cx - y = 2 \Rightarrow y = cx - 2$. Sub into the curve:

$$x^2 + x(cx - 2) = 4$$
$$x^2 + cx^2 - 2x - 4 = 0$$
$$(1 + c)x^2 - 2x - 4 = 0$$

Two points: $b^2 - 4ac > 0$

$$(-2)^2 - 4(1 + c)(-4) > 0$$
$$4 + 16 + 16c > 0$$
$$16c > -20$$
$$c > -\tfrac{5}{4}$$

When $c = -1$, the coefficient of x^2 is 0: the equation becomes $-2x - 4 = 0$, which has only one root, so the line meets the curve at one point only.

$\boxed{c > -\frac{5}{4},\ c \neq -1}$

Tangent: $b^2 - 4ac = 0 \Rightarrow 20 + 16c = 0$, $\boxed{c = -\frac{5}{4}}$

23 $px^2 + 6x + p = 8$

$px^2 + 6x + p - 8 = 0$

Lies entirely above: no point in common ($b^2 - 4ac < 0$) and the curve is U-shaped ($a > 0$).

$$6^2 - 4(p)(p - 8) < 0 \quad \text{and} \quad p > 0$$
$$36 - 4p^2 + 32p < 0$$
$$p^2 - 8p - 9 > 0$$
$$(p - 9)(p + 1) > 0$$



$p < -1$ (rejected) or $p > 9$

Since $p > 0$: $\boxed{p > 9}$

24 $y + px = 2p \Rightarrow y = 2p - px$

$$(p + 3)x^2 + px - 3 = 2p - px$$
$$(p + 3)x^2 + 2px - (2p + 3) = 0$$

(a quadratic equation, since $p + 3 \neq 0$). To show: $b^2 - 4ac > 0$

$$\begin{aligned} b^2 - 4ac &= (2p)^2 - 4(p + 3)\big(-(2p + 3)\big) \\ &= 4p^2 + 4(2p^2 + 9p + 9) \\ &= 12p^2 + 36p + 36 \\ &= 12\big(p^2 + 3p + 3\big) \\ &= 12\Big[\big(p + \tfrac{3}{2}\big)^2 - \big(\tfrac{3}{2}\big)^2 + 3\Big] \\ &= 12\Big[\big(p + \tfrac{3}{2}\big)^2 + \tfrac{3}{4}\Big] \\ &= 12\big(p + \tfrac{3}{2}\big)^2 + 9 \end{aligned}$$

Since $\big(p + \frac{3}{2}\big)^2 \geq 0$ for all real values of p,

$$12\big(p + \tfrac{3}{2}\big)^2 \geq 0$$
$$12\big(p + \tfrac{3}{2}\big)^2 + 9 \geq 9$$

$\therefore\ b^2 - 4ac > 0$: the line meets the curve at two different points for all real values of p, $p \neq -3$. (Shown)

25 $mx + m + 3 = -x^2 + 4x - 1$

$$x^2 + mx - 4x + m + 4 = 0$$
$$x^2 + (m - 4)x + (m + 4) = 0$$

The curve is n-shaped (the coefficient of x^2 is $-1 < 0$), so it lies entirely below the line when they do not meet: $b^2 - 4ac < 0$.

$$(m - 4)^2 - 4(1)(m + 4) < 0$$
$$m^2 - 8m + 16 - 4m - 16 < 0$$
$$m^2 - 12m < 0$$
$$m(m - 12) < 0$$



$\boxed{0 < m < 12}$

26 $px^2 + 8x + p - 6 \leq 0$ for all real values of x

Never positive: $b^2 - 4ac \leq 0$ and $a < 0$

$$8^2 - 4(p)(p - 6) \leq 0 \quad \text{and} \quad p < 0$$
$$64 - 4p^2 + 24p \leq 0$$
$$p^2 - 6p - 16 \geq 0$$
$$(p - 8)(p + 2) \geq 0$$



$p \leq -2$ or $p \geq 8$ (rejected)

$p \geq 8$ is rejected because the coefficient of x^2 must be negative: with $p > 0$ the curve is U-shaped and takes positive values.

$\boxed{p \leq -2}$

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