Simultaneous Linear Equations

Elimination Method, Substitution Method, Intermediate Questions, Word Problems

Secondary 2 — G3 Mathematics

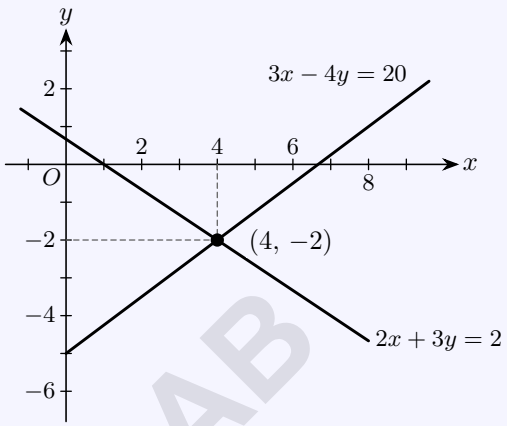
Notes

What it means to solve a pair of simultaneous equations

A pair of equations in two unknowns, such as

$$3x - 4y = 20 \qquad \text{and} \qquad 2x + 3y = 2,$$

is called a pair of **simultaneous equations**: both equations have to hold at the same time. Solving them means finding the values of x and y that make *both* equations true.



Each equation has a straight-line graph, and every point on that line satisfies its equation. So the solution is the one point that lies on both lines, the point where they cross. These two lines cross at $(4, -2)$, so the solution is $x = 4$, $y = -2$.

Reading a solution off a graph takes time and may not be exact, so we solve simultaneous equations by algebra instead, using either the **elimination method** or the **substitution method**, both set out below.

The elimination method in four steps. Step 1: Number the two equations.

Step 2: Multiply the equations so that one unknown (x or y) has the same coefficient in both. Either unknown may be removed first.

Step 3: Subtract one equation from the other (or add them), so that this unknown disappears. Solve the equation that is left.

Step 4: Substitute this value into either of the original equations to find the other unknown.

Add or subtract? Once the terms to be removed have the same coefficient, check their signs. Terms with the **same sign** cancel when the equations are **subtracted**; terms with **opposite signs** cancel when the equations are **added**. In the figure, Eqn 3 and Eqn 4 come from the pair at the start of the notes, $3x - 4y = 20$ (Eqn 1) and $2x + 3y = 2$ (Eqn 2): on the left they are (Eqn 1) $\times 2$ and (Eqn 2) $\times 3$, and on the right (Eqn 1) $\times 3$ and (Eqn 2) $\times 4$.

Same signs: subtract

(Eqn 3) − (Eqn 4)

$$\begin{array}{rrcrl} & 6x & -8y & = & 40 \quad \text{Eqn 3} \\ - & 6x & +9y & = & 6 \quad \text{Eqn 4} \\ \hline & & -17y & = & 34 \end{array}$$

$6x - 6x = 0$, so x is eliminated

Opposite signs: add

(Eqn 3) + (Eqn 4)

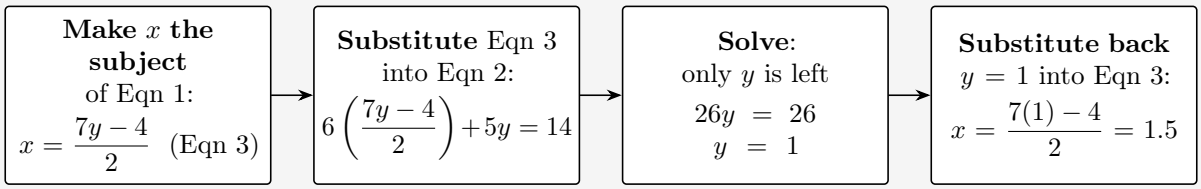
$$\begin{array}{rrcrl} & 9x & -12y & = & 60 \quad \text{Eqn 3} \\ + & 8x & +12y & = & 8 \quad \text{Eqn 4} \\ \hline & 17x & & = & 68 \end{array}$$

$-12y + 12y = 0$, so y is eliminated

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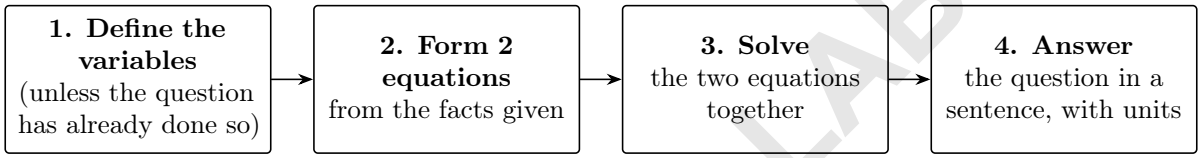
The substitution method in four steps, shown for the equations $-2x + 7y = 4$ (Eqn 1) and $6x + 5y = 14$ (Eqn 2). Once one unknown is the subject of one equation, substituting it into the *other* equation leaves an equation with only one unknown in it.



Tidy each equation into the form $Ax + By = C$ first. Only then start eliminating or substituting.

When the two equations are given as one statement, $P = Q = R$, split the statement into two equations, such as $P = Q$ and $Q = R$. It is fine for the middle part to appear in both.

Each word problem is worked through in the same four stages.



If the tens digit of a two-digit number is x and its ones digit is y, the number is $10x + y$, not xy. Can you explain why? The number with its digits reversed is $10y + x$.

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Exercises

Section 1: Simultaneous Equations (Elimination Method)

1. Solve each pair of simultaneous equations by the elimination method.

(a) $3x + 2y = 5$
$4x + 5y = 2$

(b) $8x + 3y = 17$
$4x - 5y = 41$

2. Solve each pair of simultaneous equations by the elimination method.

(a) $5x - 6y = -3$
$-4x + 7y = -2$

(b) $5x + 6y = -13$
$-6x + 5y = -21$

3. Use elimination to solve each pair of simultaneous equations.

(a) $3a + 5b = 5\frac{1}{4}$
$5a + 3b = 4\frac{3}{4}$

(b) $3x + 5y = -4$
$-2x + 5y = 11$

(c) $-3x + 2y = 0$
$6x + 7y = 11$

Section 2: Simultaneous Equations (Substitution Method)

4. Solve each pair of simultaneous equations by the substitution method.

(a) $2x + 5y = 2$
$3x + 7y = 4$

(b) $6a - 5b = 28$
$2a + 3b = 0$

(c) $4y = 3x$
$7x - 12y + 4 = 0$

5. Use substitution to solve each pair of simultaneous equations.

(a) $x + 4y = -5$
$3x - 2y = 27$

(b) $2x - 5y = -11$
$4x + y = 11$

(c) $y = 2x - 10$
$5y - 4x = -29$

6. Solve each pair of simultaneous equations by substitution.

(a) $2x = 3y + 8$
$4y + 3x = 3\frac{1}{2}$

(b) $\frac{x}{2} + \frac{y}{5} = 4$
$3x - y = 2$

(c) $\frac{1}{2}x + 2y + 4 = 0$
$\frac{1}{4}x + 3y + 4 = 0$

Section 3: Intermediate Questions

7. Solve each of the following by any suitable algebraic method.

(a) $3(x - 2) + y = 10 - 3y$
$6x - 5y = -59$

(b) $\frac{1}{5}(x + 4) = \frac{1}{2}(y - 1)$
$3(2x - y) + 4 = 2y + 15$

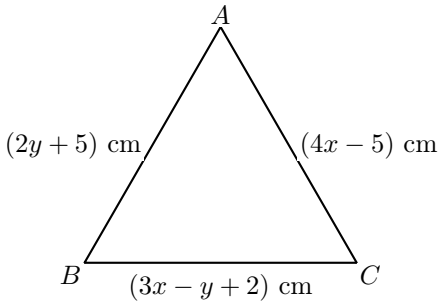
(c) $5x - 2y = 3x - y = 4$

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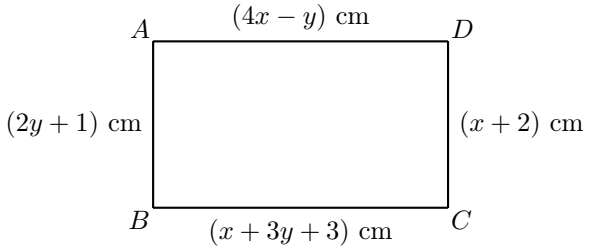
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8. The sides of each figure are given in centimetres. Find the perimeter of each figure.

(a) ABC is an equilateral triangle.



(b) $ABCD$ is a rectangle.



9. By writing $m = \dfrac{1}{x}$ and $n = \dfrac{1}{y}$, solve the simultaneous equations

$$\frac{1}{x} + \frac{2}{y} - 1 = 0, \qquad \frac{2}{x} + \frac{3}{y} = 4.$$

Section 4: Applications to Real-world Context (Word Problems)

10. Ravi pays exactly \$420 for a tablet, using only ten-dollar and fifty-dollar notes. He hands over 18 notes in all. Using an algebraic method, find how many notes of each kind he uses.

11. Four years ago, Marcus was five times as old as his brother Ken. In eight years' time, Marcus will be twice as old as Ken. Using an algebraic method, find their present ages.

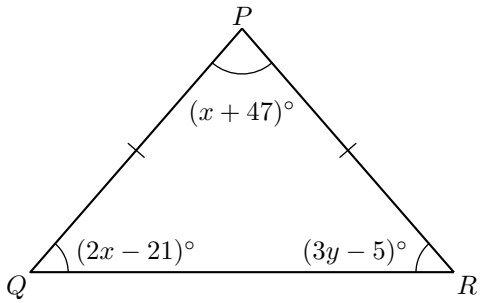
12. A two-digit number is twice the sum of its digits. When its digits are reversed, the number increases by 63. Using an algebraic method, find the number.

Supplementary Word Problems

13. In the diagram, triangle PQR is an isosceles triangle with $PQ = PR$.

(a) Write down two simultaneous equations in terms of x and y.

(b) Solve the simultaneous equations to find the size of $\angle PRQ$.



14. The sum of two integers is three times their difference. Three times the smaller integer exceeds the larger integer by 4. Find the two integers.

15. The first four terms of a sequence are 5, 9, 13 and 17. The sum of the first n terms of this sequence is given by $an^2 + bn$.

(a) When $n = 1$, the sum is the first term alone, so $a + b = 5$. By taking $n = 2$, show that $4a + 2b = 14$.

(b) Solve the simultaneous equations

$$a + b = 5, \qquad 4a + 2b = 14.$$

(c) Hence find

(i) the sum of the first 10 terms of this sequence,

(ii) the 10th term of this sequence.

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Solutions to Exercises

Section 1 — Elimination method (Questions 1–3)

1(a) $3x + 2y = 5$ (1)
$4x + 5y = 2$ (2)
(1) × 5: $15x + 10y = 25$ (3)
(2) × 2: $8x + 10y = 4$ (4)
(3) − (4):
$(15x + 10y) - (8x + 10y) = 25 - 4$
$7x = 21, \quad x = 3$
Sub $x = 3$ into (1):
$3(3) + 2y = 5, \quad 2y = -4, \quad y = -2$
$\therefore \boxed{x = 3,\ y = -2}$

1(b) $8x + 3y = 17$ (1)
$4x - 5y = 41$ (2)
(2) × 2: $8x - 10y = 82$ (3)
(1) − (3):
$(8x + 3y) - (8x - 10y) = 17 - 82$
$13y = -65, \quad y = -5$
Sub $y = -5$ into (2):
$4x - 5(-5) = 41, \quad 4x = 16, \quad x = 4$
$\therefore \boxed{x = 4,\ y = -5}$

2(a) $5x - 6y = -3$ (1)
$-4x + 7y = -2$ (2)
(1) × 4: $20x - 24y = -12$ (3)
(2) × 5: $-20x + 35y = -10$ (4)
Opposite signs, so add.
(3) + (4):
$(20x - 24y) + (-20x + 35y) = -12 + (-10)$
$11y = -22, \quad y = -2$
Sub $y = -2$ into (1):
$5x - 6(-2) = -3, \quad 5x = -15, \quad x = -3$
$\therefore \boxed{x = -3,\ y = -2}$

2(b) $5x + 6y = -13$ (1)
$-6x + 5y = -21$ (2)
(1) × 6: $30x + 36y = -78$ (3)
(2) × 5: $-30x + 25y = -105$ (4)
(3) + (4):
$(30x + 36y) + (-30x + 25y) = -78 + (-105)$
$61y = -183, \quad y = -3$
Sub $y = -3$ into (1):
$5x + 6(-3) = -13, \quad 5x = 5, \quad x = 1$
$\therefore \boxed{x = 1,\ y = -3}$

3(a) $3a + 5b = 5.25$ (1)
$5a + 3b = 4.75$ (2)
(1) × 5: $15a + 25b = 26.25$ (3)
(2) × 3: $15a + 9b = 14.25$ (4)
(3) − (4):
$(15a + 25b) - (15a + 9b) = 26.25 - 14.25$
$16b = 12, \quad b = 0.75$
Sub $b = 0.75$ into (1):
$3a + 5(0.75) = 5.25, \quad 3a = 1.5, \quad a = 0.5$
$\therefore \boxed{a = 0.5,\ b = 0.75}$

3(b) $3x + 5y = -4$ (1)
$-2x + 5y = 11$ (2)
(1) − (2):
$(3x + 5y) - (-2x + 5y) = -4 - 11$
$5x = -15, \quad x = -3$
Sub $x = -3$ into (1):
$3(-3) + 5y = -4, \quad 5y = 5, \quad y = 1$
$\therefore \boxed{x = -3,\ y = 1}$

3(c) $-3x + 2y = 0$ (1)
$6x + 7y = 11$ (2)
(1) × 2: $-6x + 4y = 0$ (3)
(3) + (2):
$(-6x + 4y) + (6x + 7y) = 0 + 11$
$11y = 11, \quad y = 1$
Sub $y = 1$ into (1):
$-3x + 2(1) = 0, \quad -3x = -2, \quad x = \frac{2}{3}$
$\therefore \boxed{x = \frac{2}{3},\ y = 1}$

Section 2 — Substitution method (Questions 4–6)

4(a) $2x + 5y = 2$ (1)
$3x + 7y = 4$ (2)
From (1):
$2x = 2 - 5y, \quad x = \dfrac{2 - 5y}{2}$ (3)
Sub (3) into (2):
$3\left(\dfrac{2 - 5y}{2}\right) + 7y = 4$
$3(2 - 5y) + 14y = 8$
$6 - 15y + 14y = 8, \quad -y = 2, \quad y = -2$
Sub $y = -2$ into (3):
$x = \dfrac{2 - 5(-2)}{2} = 6$
$\therefore \boxed{x = 6,\ y = -2}$

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4(b) $6a - 5b = 28$ (1)

$2a + 3b = 0$ (2)

From (2):

$3b = -2a, \quad b = -\dfrac{2a}{3}$ (3)

Sub (3) into (1):

$6a - 5\left(-\dfrac{2a}{3}\right) = 28$

$6a + \dfrac{10a}{3} = 28$

$18a + 10a = 84, \quad 28a = 84, \quad a = 3$

Sub $a = 3$ into (3):

$b = -\dfrac{2(3)}{3} = -2$

$\therefore \boxed{a = 3,\ b = -2}$

4(c) $4y = 3x$ (1)

$7x - 12y + 4 = 0$ (2)

From (1):

$y = \dfrac{3x}{4}$ (3)

Sub (3) into (2):

$7x - 12\left(\dfrac{3x}{4}\right) + 4 = 0$

$7x - 9x + 4 = 0, \quad -2x = -4, \quad x = 2$

Sub $x = 2$ into (3):

$y = \dfrac{3(2)}{4} = 1.5$

$\therefore \boxed{x = 2,\ y = 1.5}$

5(a) $x + 4y = -5$ (1)

$3x - 2y = 27$ (2)

From (1):

$x = -5 - 4y$ (3)

Sub (3) into (2):

$3(-5 - 4y) - 2y = 27$

$-15 - 12y - 2y = 27$

$-14y = 42, \quad y = -3$

Sub $y = -3$ into (3):

$x = -5 - 4(-3) = 7$

$\therefore \boxed{x = 7,\ y = -3}$

5(b) $2x - 5y = -11$ (1)

$4x + y = 11$ (2)

From (2):

$y = 11 - 4x$ (3)

Sub (3) into (1):

$2x - 5(11 - 4x) = -11$

$2x - 55 + 20x = -11, \quad 22x = 44, \quad x = 2$

Sub $x = 2$ into (3):

$y = 11 - 4(2) = 3$

$\therefore \boxed{x = 2,\ y = 3}$

5(c) $y = 2x - 10$ (1)

$5y - 4x = -29$ (2)

Sub (1) into (2):

$5(2x - 10) - 4x = -29$

$10x - 50 - 4x = -29, \quad 6x = 21, \quad x = 3.5$

Sub $x = 3.5$ into (1):

$y = 2(3.5) - 10 = -3$

$\therefore \boxed{x = 3.5,\ y = -3}$

6(a) $2x = 3y + 8$ (1)

$4y + 3x = 3\frac{1}{2}$ (2)

From (1):

$x = \dfrac{3y + 8}{2}$ (3)

Sub (3) into (2):

$4y + 3\left(\dfrac{3y + 8}{2}\right) = 3\frac{1}{2}$

$8y + 3(3y + 8) = 7$

$8y + 9y + 24 = 7, \quad 17y = -17, \quad y = -1$

Sub $y = -1$ into (3):

$x = \dfrac{3(-1) + 8}{2} = 2\frac{1}{2}$

$\therefore \boxed{x = 2\frac{1}{2},\ y = -1}$

6(b) $\dfrac{x}{2} + \dfrac{y}{5} = 4$ (1)

$3x - y = 2$ (2)

From (2):

$y = 3x - 2$ (3)

Sub (3) into (1):

$\dfrac{x}{2} + \dfrac{3x - 2}{5} = 4$

$\times 10$:

$5x + 2(3x - 2) = 40$

$5x + 6x - 4 = 40, \quad 11x = 44, \quad x = 4$

Sub $x = 4$ into (3):

$y = 3(4) - 2 = 10$

$\therefore \boxed{x = 4,\ y = 10}$

6(c) $\frac{1}{2}x + 2y + 4 = 0$ (1)

$\frac{1}{4}x + 3y + 4 = 0$ (2)

From (1):

$\frac{1}{2}x = -2y - 4, \quad x = -4y - 8$ (3)

Sub (3) into (2):

$\frac{1}{4}(-4y - 8) + 3y + 4 = 0$

$-y - 2 + 3y + 4 = 0, \quad 2y = -2, \quad y = -1$

Sub $y = -1$ into (3):

$x = -4(-1) - 8 = -4$

$\therefore \boxed{x = -4,\ y = -1}$

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Section 3 — Intermediate questions (Questions 7–9)

7(a) $3(x-2)+y=10-3y$

$3x-6+y=10-3y$

$3x+4y=16$ (1)

$6x-5y=-59$ (2)

(1) $\times 2$: $6x+8y=32$ (3)

(3) $-$ (2):

$(6x+8y)-(6x-5y)=32-(-59)$

$13y=91, \quad y=7$

Sub $y=7$ into (1):

$3x+4(7)=16, \quad 3x=-12, \quad x=-4$

$\therefore$ $\boxed{x=-4,\ y=7}$

7(b) $\frac{1}{5}(x+4)=\frac{1}{2}(y-1)$

$\times 10$:

$2(x+4)=5(y-1)$

$2x+8=5y-5$

$2x-5y=-13$ (1)

$3(2x-y)+4=2y+15$

$6x-3y+4=2y+15$

$6x-5y=11$ (2)

(2) $-$ (1):

$(6x-5y)-(2x-5y)=11-(-13)$

$4x=24, \quad x=6$

Sub $x=6$ into (1):

$2(6)-5y=-13, \quad -5y=-25, \quad y=5$

$\therefore$ $\boxed{x=6,\ y=5}$

7(c) Split into 2 equations.

$5x-2y=4$ (1)

$3x-y=4$ (2)

(2) $\times 2$: $6x-2y=8$ (3)

(3) $-$ (1):

$(6x-2y)-(5x-2y)=8-4$

$x=4$

Sub $x=4$ into (2):

$3(4)-y=4, \quad -y=-8, \quad y=8$

$\therefore$ $\boxed{x=4,\ y=8}$

8(a) The sides of an equilateral triangle are all equal.

$AB=AC$:

$2y+5=4x-5$

$4x-2y=10$

$\div 2$:

$2x-y=5$ (1)

$AC=BC$:

$4x-5=3x-y+2$

$x+y=7$ (2)

(1) + (2):

$(2x-y)+(x+y)=5+7$

$3x=12, \quad x=4$

Sub $x=4$ into (2):

$4+y=7, \quad y=3$

Each side $=4(4)-5=11$ cm

Perimeter $=3\times 11=$ $\boxed{33\text{ cm}}$

8(b) Opposite sides of a rectangle are equal.

$AD=BC$:

$4x-y=x+3y+3$

$3x-4y=3$ (1)

$AB=DC$:

$2y+1=x+2$

$x-2y=-1$ (2)

(2) $\times 2$: $2x-4y=-2$ (3)

(1) $-$ (3):

$(3x-4y)-(2x-4y)=3-(-2)$

$x=5$

Sub $x=5$ into (2):

$5-2y=-1, \quad -2y=-6, \quad y=3$

$AD=4(5)-3=17$ cm, $\quad AB=2(3)+1=7$ cm

Perimeter $=2(17+7)=$ $\boxed{48\text{ cm}}$

9 With $m=\dfrac{1}{x}$ and $n=\dfrac{1}{y}$:

$m+2n-1=0$

$m+2n=1$ (1)

$2m+3n=4$ (2)

(1) $\times 2$: $2m+4n=2$ (3)

(3) $-$ (2):

$(2m+4n)-(2m+3n)=2-4$

$n=-2$

Sub $n=-2$ into (1):

$m+2(-2)=1, \quad m=5$

$x=\dfrac{1}{m}=\dfrac{1}{5}, \quad y=\dfrac{1}{n}=-\dfrac{1}{2}$

$\therefore$ $\boxed{x=\frac{1}{5},\ y=-\frac{1}{2}}$

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Section 4 — Word problems (Questions 10–12)

10 Let the number of ten-dollar notes be x and the number of fifty-dollar notes be y.

$x + y = 18$ (1)

$10x + 50y = 420$ (2)

From (1):

$x = 18 - y$ (3)

Sub (3) into (2):

$10(18 - y) + 50y = 420$

$180 - 10y + 50y = 420, \quad 40y = 240, \quad y = 6$

Sub $y = 6$ into (3):

$x = 18 - 6 = 12$

He uses $\boxed{12}$ ten-dollar notes and $\boxed{6}$ fifty-dollar notes.

11 Let Marcus's present age be x years and Ken's present age be y years.

$x - 4 = 5(y - 4)$

$x - 4 = 5y - 20$

$x - 5y = -16$ (1)

$x + 8 = 2(y + 8)$

$x + 8 = 2y + 16$

$x - 2y = 8$ (2)

(2) − (1):

$(x - 2y) - (x - 5y) = 8 - (-16)$

$3y = 24, \quad y = 8$

Sub $y = 8$ into (2):

$x - 2(8) = 8, \quad x = 24$

Marcus is $\boxed{24}$ years old and Ken is $\boxed{8}$ years old.

12 Let the tens digit be x and the ones digit be y. The number is $10x + y$.

$10x + y = 2(x + y)$

$10x + y = 2x + 2y$

$8x - y = 0$ (1)

$(10y + x) - (10x + y) = 63$

$-9x + 9y = 63$

$\div 9$:

$-x + y = 7$ (2)

(1) + (2):

$(8x - y) + (-x + y) = 0 + 7$

$7x = 7, \quad x = 1$

Sub $x = 1$ into (1):

$8(1) - y = 0, \quad y = 8$

The number is $\boxed{18}$.

Supplementary word problems (Questions 13–15)

13(a) $\angle PQR = \angle PRQ$ (base $\angle$s of isos. $\triangle$):

$2x - 21 = 3y - 5$

$\boxed{2x - 3y = 16}$ (1)

$\angle$ sum of $\triangle$:

$(x + 47) + (2x - 21) + (3y - 5) = 180$

$3x + 3y + 21 = 180$

$3x + 3y = 159$

$\div 3$:

$\boxed{x + y = 53}$ (2)

13(b) From (2):

$x = 53 - y$ (3)

Sub (3) into (1):

$2(53 - y) - 3y = 16$

$106 - 2y - 3y = 16$

$-5y = -90, \quad y = 18$

Sub $y = 18$ into (3):

$x = 53 - 18 = 35$

$\angle PRQ = (3y - 5)^\circ = (54 - 5)^\circ$

$\angle PRQ = \boxed{49^\circ}$

14 Let the larger integer be x and the smaller integer be y.

$x + y = 3(x - y)$

$x + y = 3x - 3y$

$-2x + 4y = 0$

$\div(-2)$:

$x - 2y = 0$ (1)

$3y - x = 4$

$-x + 3y = 4$ (2)

(1) + (2):

$(x - 2y) + (-x + 3y) = 0 + 4$

$y = 4$

Sub $y = 4$ into (1):

$x - 2(4) = 0, \quad x = 8$

The two integers are $\boxed{8}$ and $\boxed{4}$.

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15(a) When $n = 2$, the sum is $5 + 9 = 14$.

$a(2)^2 + b(2) = 14$

$4a + 2b = 14$ (shown)

15(b) $a + b = 5$ (1)

$4a + 2b = 14$ (2)

(1) $\times$ 2: $2a + 2b = 10$ (3)

(2) $-$ (3):

$(4a + 2b) - (2a + 2b) = 14 - 10$

$2a = 4, \quad a = 2$

Sub $a = 2$ into (1):

$2 + b = 5, \quad b = 3$

$\therefore \boxed{a = 2,\ b = 3}$

15(c) The sum of the first n terms is $2n^2 + 3n$.

(i) Sum of the first 10 terms

$= 2(10)^2 + 3(10) = 200 + 30 = \boxed{230}$

(ii) The 10th term is the sum of the first 10 terms minus the sum of the first 9 terms.

Sum of the first 9 terms

$= 2(9)^2 + 3(9) = 162 + 27 = 189$

10th term $= 230 - 189 = \boxed{41}$

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